

Chapter 1: Review of basic concepts

1.1. Operations on sets

Let A and B be sets with universal set U .

The complement of set A is the set A' with all elements in the universal set that do not belong to set A .

The intersection of sets A and B written $A \cap B$, is made up of all the elements belonging to both set A and set B .

$$A \cap B = \{x: x \in A \text{ and } x \in B\}$$

The union of sets A and B , written $A \cup B$, is made up of all the elements belonging to set A or set B .

$$A \cup B = \{x: x \in A \text{ or } x \in B\}$$

Ex: Let $U = \{1,2,3,4,5,6\}$, $A = \{1,2,3,4\}$ and $B = \{3,4,6\}$

Find :

- (i) A' (ii) $A \cap B$ (iii) $A \cup B$

Solution :

(i) $A' = \{5,6\}$

(ii) $A \cap B = \{3,4\}$

(iii) $A \cup B = \{1,2,3,4,6\}$

1.2. *1*Sets of numbers (Real Numbers and their properties)

- (i) The set of natural numbers ($\mathbb{N}$) :

$$\mathbb{N} = \{1,2,3 \dots\}$$

- (ii) The set of whole numbers (W) :

$$W = \{0,1,2,\dots\}$$

- (iii) The set of integer numbers ($\mathbb{Z}$) :

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$$

(iv) The set of rational numbers ($\mathbb{Q}$) :

$$\mathbb{Q} = \left\{ \frac{a}{b}; a, b \in \mathbb{Z}, b \neq 0 \right\}$$

$$\text{Ex : } \frac{-19}{17}, \frac{1}{3}, 5, -17, 1.37, 0, \sqrt{q}, 3.666$$

(v) The set of irrational numbers (I) :

$$\text{Ex : } \sqrt{2}, \pi, 3.2179 \dots$$

(vi) The set of real numbers ($\mathbb{R}$) :

Def : The set of rational numbers ($\mathbb{Q}$) and the set of irrational numbers (I) are called real numbers.

Ex(1) : Choose the correct answer

One of the following numbers is an irrational number :

$$\text{a) } \frac{22}{3} \quad \text{b) } \sqrt{7} \quad \text{c) } \sqrt{16} \quad \text{d) None}$$

***2*Arithmetic operations :**

Addition, subtraction, multiplication, and division.

Ex(2) : Simplify

$$\text{a. } 4 + 3 \times 2 - 7 \times 5 + 6 = 4 + 6 - 35 + 6 = -19$$

$$\text{b. } \frac{7-6}{2 \times 3+8} = \frac{7-6}{6+8} = \frac{1}{14}$$

$$\begin{aligned} \text{c. } 3[5 \times (4 - 2) - 2 \times 7] &= 3[5 \times 2 - 2 \times 7] \\ &= 3[10 - 14] \\ &= 3(-4) \\ &= -12 \end{aligned}$$

***3*Properties of Negative**

For a,b are real numbers, then :

$$\text{i. } (-a)(b) = -ab$$

$$\text{ii. } (-a)(-b) = ab$$

$$\text{iii. } \frac{a}{-b} = \frac{-a}{b}$$

$$\text{iv. } =$$

$$\text{v. } -(a + b) = -a - b$$

Ex(3) : Perform the operations

$$a. -5 + 7 - (-2) = -5 + 7 + 2 = 4$$

$$b. -(x - y) = -x + y = y - x$$

***4* Absolute values:**

Let a is real number, then :

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Ex(4) : Evaluate

$$a. |-3 + 7| = |4| = 4$$

$$b. |2 - 8| = |-6| = 6$$

$$c. |0| = 0$$

***5* Fractional properties**

For a, b, d and c are real numbers, then:

$$i. \frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$$

$$ii. \frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$$

$$iii. \frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$$

$$iv. \frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

$$v. \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} * \frac{d}{c}$$

Ex(5) : Perform and simplify

$$a. \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$$

$$b. \frac{x}{2} + \frac{3}{5} = \frac{5x+6}{10}$$

$$c. \frac{2}{3} \div \frac{4}{1} = \frac{2}{3} * \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$$

***6* Integer Exponents:**

Let a be a real number and n a natural number, then:

$$a^n = a * a * a \dots a$$

where:

n $\equiv$ exponent or power

a $\equiv$ base

Ex(1) : Evaluate

$$i. 3^4 = 3 * 3 * 3 * 3 = 81$$

- ii. $(-3)^3 = (-3)(-3)(-3) = -27$
- iii. $-3^4 = -[3 * 3 * 3 * 3] = -81$
- iv. $\left(\frac{1}{2}\right)^5 = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{32}$
- v. $6 \div 3 + 2^3 * 5 = 6 \div 3 + 8 * 5 = 2 + 8 * 5 = 2 + 40 = 42$

***7* Negative Integer Exponents :**

Let a be any nonzero real number and n a natural number, then:

$$a^{-n} = \frac{1}{a^n}, \quad (a \neq 0)$$

Ex(2) : Evaluate

- a. $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$
- b. $\frac{1}{3^{-3}} = 3^3 = 27$
- c. $\left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^2 = 2^2 = 4$
- d. $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{3^3}{2^3} = \frac{27}{8}$

Definition : Zero Exponent

Let a be any nonzero real number, then :

$$a^0 = 1, \quad (a \neq 0)$$

Ex(3) : Evaluate

$$(i) 5^0 = 1 \quad (ii) (-2)^0 = 1 \quad (iii) x^0 = 1 \quad (iv) \left(\frac{1}{2}\right)^0 = 1$$

***8* Rules of Integer Exponents :**

- i. $a^m * a^n = a^{m+n}$
- ii. $\frac{a^m}{a^n} = a^{m-n}$
- iii. $(a^m)^n = a^{mn}$
- iv. $(ab)^n = a^n * b^n$
- v. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Ex(4) : Simplify

- i. $(-2x^2y^2)(5x^3y) = -10x^5y^3$
- ii. $(2x^2yz^3)^3 = (2^3)(x^2)^3(y^3)(z^3)^3$
 $= 8x^6y^3z^9$

$$\text{iii. } \frac{25x^3y^6}{-5x^5y^4} = -5x^{-2}y^2 = \frac{-5y^2}{x^2}$$

$$\begin{aligned} \text{iv. } (3x^2z^{-4})^{-3} &= (3^{-3})(x^2)^{-3}(z^{-4})^{-3} \\ &= \frac{1}{27}x^{-6}z^{12} = \frac{z^{12}}{27x^6} \end{aligned}$$

1.3. *1*Polynomials

A polynomial in variable x is an algebraic expression of the form :

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x + a_0$$

where :

$a_n \equiv$ leading coefficient

$n \equiv$ Degree

$a_0 \equiv$ constant term

Ex(1) : Given the polynomial

$$p(x) = 4x - 9x^5 + 2, \text{ state :}$$

i. Degree=5

ii. leading coefficient=-9

iii. constant term=2

iv. Number of terms=3

v. Standard form $\Rightarrow p(x) = -9x^5 + 4x + 2$

Ex(2) : (Add and sub polynomials)

Find and simplify

$$\text{i. } (5x^2 - 2x + 3) + (3x^3 - 4x^2 + 7)$$

Solution :

$$5x^2 - 2x + 3 + 3x^3 - 4x^2 + 7 = 3x^3 + x^2 - 2x + 10$$

$$\text{ii. } (2x^2 - 3x - 4) - 2(x^2 - 3x - 3)$$

Solution :

$$2x^2 - 3x - 4 - 2x^2 + 6x + 6 = 3x + 2$$

Ex(3) : (Multiplying polynomials)

Find and simplify

$$\text{i. } 5x^2(3x^5 - x^3 + 7x - 4) = 15x^7 - 5x^5 + 35x^3 - 20x^2$$

$$\begin{aligned}\text{ii. } (3x + 1)(2x - 5) &= 6x^2 - 15x + 2x - 5 \\ &= 6x^2 - 13x - 5\end{aligned}$$

$$\text{iii. } (2x - 5)(x^2 - 2x + 3)$$

***2* Special product :**

$$\text{i. } (a + b)(a - b) = a^2 - b^2 \quad (\text{Difference between two squares})$$

$$\text{ii. } (a + b)^2 = a^2 + 2ab + b^2 \quad \text{(Perfect square)}$$

$$\text{iii. } (a - b)^2 = a^2 - 2ab + b^2$$

Ex(4) : Find

$$\text{i. } (4x + 3)(4x - 3) = (4x)^2 - 3^2 = 16x^2 - 9$$

$$\begin{aligned}\text{ii. } (2x - 1)^2 &= (2x)^2 - 2(2x)(1) + 1^2 \\ &= 4x^2 - 4x + 1\end{aligned}$$

$$\begin{aligned}\text{iii. } (3 + 2y)^2 &= 3^2 + 2(3)(2y) + (2y)^2 \\ &= 9 + 12y + 4y^2\end{aligned}$$

1.4. Factoring polynomials :

***1* Factoring special products :**

$$\text{i. } a^2 - b^2 = (a + b)(a - b) \quad (\text{Difference between two squares})$$

$$\text{ii. } a^2 + 2ab + b^2 = (a + b)^2 \quad \text{(Perfect square)}$$

$$\text{iii. } a^2 - 2ab + b^2 = (a - b)^2$$

Ex(1) : Factor

$$\text{i. } x^2 - 9 = (x - 3)(x + 3)$$

$$\text{ii. } 4x^2 - 25 = (2x + 5)(2x - 5)$$

$$\text{iii. } 16 - 9x^2 = (4 - 3x)(4 + 3x)$$

$$\text{iv. } x^2 + 6x + 9 = x^2 + 6x + 3^2 = (x + 3)^2$$

$$\text{v. } 4x^2 - 4x + 1 = (2x)^2 - 4x + 1^2 = (2x - 1)^2$$

***2* Factoring by extracting the common factor :**

Ex(2) : Factor

$$\text{i. } 6x^5 - 18x^4 = 6x^4(x - 3)$$

$$\text{ii. } 9y^5 + y^2 = y^2(9y^3 + 1)$$

$$\text{iii. } 6x^2t + 8xt + 12t = 2t(3x^2 + 4x + 6)$$

$$\text{iv. } 6x^5 - 10x^4 - 8x^3 + 12x^2 = 2x^2(3x^3 - 5x^2 - 4x + 6)$$

***3*Factoring $ax^2 + bx + c$, if $a = 1$**

$$a \Rightarrow x^2 + bx + c = (x + \blacksquare)(x + \blacksquare)$$

Ex(3) : factor

$$\text{i. } x^2 + 10x + 9 = (x + 9)(x + 1)$$

$$\text{ii. } x^2 - 3x - 28 = (x - 7)(x + 4)$$

$$\text{iii. } x^2 - 5x + 6 = (x - 2)(x - 3)$$

$$\text{iv. } x^2 - x - 6 = (x - 3)(x + 2)$$

1.5. Rational Expressions :

Ex(1) : (finding the domain)

Find the domain of rational expressions:

$$\text{(i) } \frac{x+6}{x+2} \quad \text{(ii) } \frac{x+4}{(x+3)(x-1)}$$

Solution :

$$\text{(i) } x + 2 = 0 \Rightarrow x = -2$$

$$\text{Domain} = \{x : x \neq -2\}$$

$$\text{or Domain} = \mathbb{R} - \{-2\}$$

$$\text{(ii) } (x + 3)(x - 1) = 0$$

$$\Rightarrow (x + 3) = 0 \text{ or } (x - 1) = 0$$

$$\Rightarrow x = -3 \text{ or } x = 1$$

$$\text{Domain} = \{x : x \neq -3 \text{ and } x \neq 1\} \text{ or}$$

$$= \mathbb{R} - \{-3, 1\}$$

Ex(2) : (Multipling and dividing rational Expressions)

Simplify

$$\text{i. } \frac{2y^2}{9} \times \frac{27}{8y^5} = \frac{(2y^2)(27)}{(9)(8y^5)} = \frac{3y^2}{4y^5} = \frac{3}{4y^3}$$

$$\text{ii. } \frac{5m+25}{10} \div \frac{6m+30}{12}$$

$$\begin{aligned}
&= \frac{5m+25}{10} \times \frac{12}{6m+30} \\
&= \frac{5(m+5)}{10} \times \frac{12}{6(m+5)} = 1
\end{aligned}$$

Ex(3) : (Add and sub rational expressions)

Simplify

$$\begin{aligned}
\text{i. } \quad & \frac{\frac{6}{1} - \frac{5}{k}}{\frac{1}{1} + \frac{5}{k}} = \frac{\frac{6k-5}{k}}{\frac{k+5}{k}} = \frac{6k-5}{k+5} \\
\text{ii. } \quad & \frac{5}{9x^2} + \frac{1}{6x} = \frac{30x+9x^2}{54x^3} \\
&= \frac{3x(10+3x)}{54x^3} = \frac{10+3x}{18x^2}
\end{aligned}$$

1.6. Rational Exponents :

Def : let m be any integer, n be any positive integer, and a be any real number, then $a^{m/n}$ is called rational exponent.

Ex : Evaluate

$$\begin{aligned}
\text{i. } \quad & 125^{2/3} = \left(\frac{3}{5}\right)^{2/3} = 5^2 = 25 \\
\text{ii. } \quad & 32^{7/5} = \left(\frac{5}{2}\right)^{7/5} = 2^7 = 128 \\
\text{iii. } \quad & -81^{3/2} = -\left[\left(\frac{3}{9}\right)^{3/3}\right] = -[9^3] = -729 \\
\text{iv. } \quad & 16^{-3/4} = \left(\frac{4}{2}\right)^{-3/4} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8} \\
\text{v. } \quad & (-4)^{5/2} \text{ not real number}
\end{aligned}$$

1.7. Radical Expressions

Def : Let a is real number, n is a positive integer, then

$$\sqrt[n]{a} = a^{1/n}$$

if $n = 2$, we write $\sqrt{a} = a^{1/2}$

$$n = 3 \Rightarrow \sqrt[3]{a} = a^{1/3}$$

⋮

etc

Ex(1) : Evaluate

$$\begin{aligned}
\text{i. } \quad & \sqrt[4]{16} = 16^{1/4} = \left(\frac{4}{2}\right)^{1/4} = 2^1 = 2 \\
\text{ii. } \quad & \sqrt[5]{-32} = -32^{1/5} = ((-2)^5)^{1/5} = -2
\end{aligned}$$

iii. $\sqrt[4]{-16}$ not real number

Ex(2) : write each expression in exponential form

i. $\sqrt[4]{x^5} = (x^5)^{1/4} = x^{5/4}$

ii. $10(\sqrt[5]{z})^2 = 10((z)^{\frac{1}{5}})^2 = 10(z)^{\frac{2}{5}}$

iii. $\sqrt{p^2 + z} = (p^2 + z)^{\frac{1}{2}}$

Rules for radicals :

i. $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ (product rule)

ii. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ (quotient rule)

iii. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ (power rule)

Ex(3) : Simplify

i. $\sqrt[3]{m}\sqrt[3]{m^2} = \sqrt[3]{m^3} = (m^{\frac{3}{3}})^{\frac{1}{3}} = m$

ii. $\sqrt{\frac{7}{64}} = \frac{\sqrt{7}}{\sqrt{64}} = \frac{\sqrt{7}}{8}$

iii. $\sqrt[4]{\sqrt{3}} = \sqrt[8]{3} = 3^{\frac{1}{8}}$

Ex(4) : Find

$(\sqrt{2} + 3)(\sqrt{8} - 5)$

Solution :

$$\begin{aligned} (\sqrt{2} + 3)(\sqrt{8} - 5) &= \sqrt{16} - 5\sqrt{2} + 3\sqrt{8} - 15 \\ &= 4 - 5\sqrt{2} + 3\sqrt{8} - 15 \\ &= -11 - 5\sqrt{2} + 3\sqrt{8} \quad (\sqrt{8} = \sqrt{4 * 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}) \\ &= -11 - 5\sqrt{2} + 3(2\sqrt{2}) \\ &= -11 - 5\sqrt{2} + 6\sqrt{2} \\ &= -11 + \sqrt{2} \end{aligned}$$

Lecture (1)

Chapter Zero

Precalculus

1.2. Review of Elementary Mathematics:

In this section we review the terminology, notation, and formulas of elementary mathematics.

Sets

A set is a collection of distinct objects. The objects in a set are called the elements or members of the set. We will denote sets by capital letters $A, B, C, \dots$ and use lowercase letters $a, b, c, \dots$ to denote the elements

Notions and Notation

the object x is in the set A	$x \in A$
the object x is not in the set A	$x \notin A$
the set of all x which satisfy property P	$\{x: P\}$
$(\{x: x^2 = 9\} = \{-3, 3\})$	
A is a subset of B , A is contained in B	$A \subseteq B$
B contains A	$B \supseteq A$
the union of A and B	$A \cup B$
$(A \cup B = \{x: x \in A \text{ or } x \in B\})$	
the intersection of A and B	$A \cap B$
$(A \cap B = \{x: x \in A \text{ and } x \in B\})$	
the empty set	$\emptyset$

These are the only notions from set theory that you will need at this point.

Real Numbers

Classification

positive integers [†]	$1, 2, 3, \dots$
integers	$0, 1, -1, 2, -2, 3, -3, \dots$
rational numbers	p/q , with p, q integers, $q \neq 0$; for example, $5/2, -19/7, -4/1 = -4$
irrational numbers	real numbers that are not rational;

for example $\sqrt{2}, \sqrt[3]{7}, \pi$

Decimal Representation

Each real number can be expressed as a decimal. To express a rational number p/q as a decimal, we divide the denominator q into the numerator p . The resulting decimal either terminates or repeats:

$$\frac{3}{5} = 0.6, \quad \frac{27}{20} = 1.35, \quad \frac{43}{8} = 5.375$$

are terminating decimals,

$$\frac{2}{3} = 0.6666 \dots = 0.\overline{6}, \quad \frac{15}{11} = 1.363636 \dots = 1.\overline{36},$$

$$\frac{116}{37} = 3.135135 \dots = 3.\overline{135}$$

are repeating decimals. (The bar over the sequence of digits indicates that the sequence repeats indefinitely.) The converse is also true; namely, every terminating or repeating decimal represents a rational number.

The decimal expansion of an irrational number can neither terminate nor repeat.

The expansions

$$\sqrt{2} = 1.414213562 \dots \quad \text{and} \quad \pi = 3.141592653 \dots$$

do not terminate and do not develop any repeating pattern.

If we stop the decimal expansion of a given number at a certain decimal place, then the result is a rational number that approximates the given number. For instance, $1.414 = 1414/1000$ is a rational number approximation to $\sqrt{2}$ and $3.14 = 314/100$ is a rational number approximation to π . More accurate approximations can be obtained by using more decimal places from the expansions.

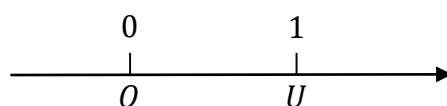


Figure 1.2.1

The Number Line (Coordinate Line, Real Line)

On a horizontal line we choose a point O . We call this point the origin and assign to it coordinate 0. Now we choose a point U to the right of O and assign to it coordinate 1. See Figure 1.2.1. The distance between O and U determines a scale (a unit length). We go on as follows: the point a units to the right of O is assigned coordinate a ; the point a units to the left of O is assigned coordinate $-a$. In this manner we establish a one-to-one correspondence between the points of a line and the numbers of the real number system. Figure 1.2.2 shows some real numbers represented as points on the number line. Positive numbers appear to the right of 0, negative numbers to the left of 0.

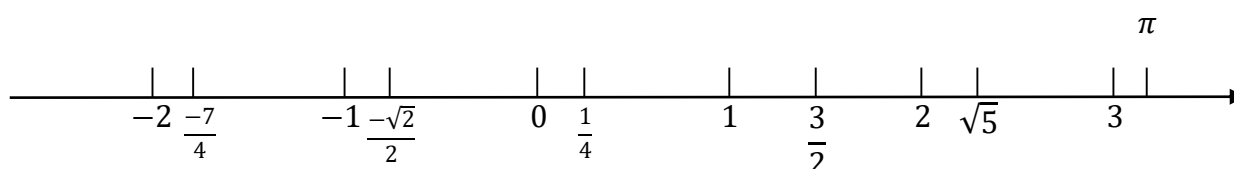


Figure 1.2.2

Order properties

- (i) $a < b, b < a, \text{ or } a = b.$
- (ii) If $a < b$ and $b < c$, then $a < c$.
- (iii) If $a < b$, then $a + c < b + c$ for all real numbers c .
- (iv) If $a < b$ and $c > 0$, then $ac < bc$.
- (v) If $a < b$ and $c < 0$, then $ac > bc$.

Absolute Value

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

Other characterizations $|a| = \max \{a, -a\}; |a| = \sqrt{a^2}.$

geometric interpretation $|a| = \text{distance between } a \text{ and } 0;$

$$|a - c| = \text{distance between } a \text{ and } c.$$

properties

(i) $|a| = 0 \text{ if and only if } a = 0.$ [†]

(ii) $|-a| = |a|.$

(iii) $|ab| = |a||b|.$

(iv) $|a + b| \leq |a| + |b|.$ (the triangle inequality)^{††}

$$(v) ||a| - |b|| \leq |a - b|. \quad (\text{a variant of the triangle inequality})$$

$$(vi) |a|^2 = |a^2| = a^2.$$

Intervals

Suppose that $a < b$. The open interval (a, b) is the set of all numbers between a and b :

$$(a, b) = \{x: a < x < b\}.$$


The closed interval $[a, b]$ is the open interval (a, b) together with the end points a and b :

$$[a, b] = \{x: a \leq x \leq b\}$$


"There are seven other types of intervals:

$$(a, b] = \{x: a < x \leq b\},$$

$$[a, b) = \{x: a \leq x < b\},$$

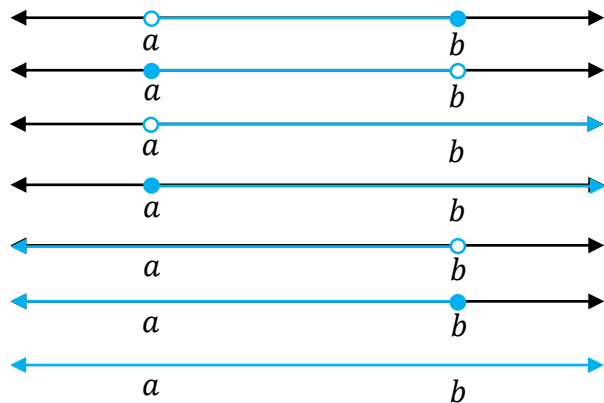
$$(a, \infty) = \{x: a < x\},$$

$$[a, \infty) = \{x: a \leq x\},$$

$$(-\infty, b) = \{x: x < b\},$$

$$(-\infty, b] = \{x: x \leq b\},$$

$$(-\infty, \infty) = \text{the set of real numbers}.$$



Interval notation is easy to remember: we use a square bracket to include an endpoint and a parenthesis to exclude it. On a number line, inclusion is indicated by a solid dot, exclusion by an open dot. The symbols ∞ and $-\infty$, read “infinity” and “negative infinity” (or “minus infinity”), do not represent real numbers. In the intervals listed above, the symbol ∞ is used to indicate that the interval extends indefinitely in the positive direction; the symbol $-\infty$ is used to indicate that the interval extends indefinitely in the negative direction.

Open and Closed

Any interval that contains no endpoints is called open: (a, b) , (a, ∞) , $(-\infty, b)$, $(-\infty, \infty)$ are open. Any interval that contains each of its endpoints (there may be one or two) is called closed: $[a, b]$, $[a, \infty)$, $(-\infty, b]$ are closed. The intervals $(a, b]$ and $[a, b)$ are called *half-open* (*half-closed*): $(a, b]$ is open on the left and closed on the right; $[a, b)$ is closed on the left and open on the right. Points of an interval that are not endpoints are called interior points of the interval.

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$a^2 - b^2 = (a - b)(a + b)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3)$$

More generally:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$$

Quadratic Equations

The roots of a quadratic equation

$$ax^2 + bx + c = 0 \quad \text{with} \quad a \neq 0$$

are given by the general quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac > 0$, the equation has two real roots; if $b^2 - 4ac = 0$, the equation has one real root; if $b^2 - 4ac < 0$, the equation has no real roots, but it has two complex roots.

Exercises

Exercises 11 – 16. Replace the symbol by $<$ or $=$ to make the statement true.

11. $\frac{3}{4} * 0.75$.

12. $0.33 * \frac{1}{3}$.

13. $\sqrt{2} + 1.414$.

14. $4 * \sqrt{16}$.

15. $-\frac{2}{7} * -0.285714$.

16. $\pi * \frac{22}{7}$.

Exercises 24 – 33. Indicate on a number line the numbers x that satisfy the condition

24. $x \geq 3$

25. $x \leq -\frac{3}{2}$.

26. $-2 \leq x \leq 3$.

29. $|x| \leq 0$.

30. $x^2 \geq 0$.

31. $|x - 4| \leq 2$.

32. $|x + 1| > 3$.

33. $|x + 3| \leq 0$.

Exercises 34 – 40 Sketch the set on a number line

34. $[3, \infty)$

35. $(-\infty, 2)$

36. $(-4, 3]$

37. $[-2, 3] \cup [1, 5]$

38. $[-3, \frac{3}{2}) \cap (\frac{3}{2}, \frac{5}{2}]$.

39. $(-\infty, -1) \cup (-2, \infty)$

40. $(-\infty, 2) \cap [3, \infty)$

Exercises 57 – 64. Find the real roots of the equation.

57. $x^2 - x - 2 = 0$.

58. $x^2 - 9 = 0$.

59. $x^2 - 6x + 9 = 0$.

60. $2x^2 - 5x - 3 = 0$.

61. $x^2 - 2x + 2 = 0$.

62. $x^2 + 8x + 16 = 0$.

63. $x^2 + 4x + 13 = 0$.

64. $x^2 - 2x + 5 = 0$.

Lecture (2)

Chapter Zero

Precalculus

1.3. Review of Inequalities:

All our work with inequalities is based on the order properties of the real numbers given in Section I -2. In this section we work with the type if inequalities that arise frequently in calculus, inequalities that involve a variable.

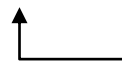
To solve an inequality in x is to find the numbers x that satisfy the inequality. These numbers constitute a set, called the solution set of the inequality.

We solve inequalities much as we solve an equation, but there is one Important difference. We can maintain an inequality by adding the same number to both sides, or by subtracting the same number from both sides, or by multiplying or dividing both sides by the same positive number. But if we multiply or divide by a negative number, then the inequality is reversed:

$$x - 2 < 4 \quad \text{gives} \quad x < 6, \quad x + 2 < 4 \quad \text{gives} \quad x < 2,$$

$$\frac{1}{2}x < 4 \quad \text{gives} \quad x < 8,$$

but $-\frac{1}{2}x < 4 \quad \text{gives} \quad x > -8.$

 Note, the inequality is reversed

Example 1 Solve the inequality

$$-3(4 - x) \leq 12.$$

SOLUTION Multiplying both sides of the inequality by $-\frac{1}{3}$, we have

$$4 - x \geq -4. \quad (\text{the inequality has been reversed})$$

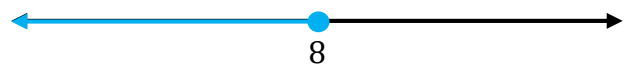
Subtracting 4, we get

$$-x \geq -8.$$

To isolate x , we multiply by -1 . This gives

$$x \leq 8. \quad (\text{the inequality has been reversed again})$$

The solution set is the interval $(-\infty, 8]$.



There are generally several ways to solve a given inequality. For example, the last inequality could have been solved as follows:

$$-3(4 - x) \leq 12,$$

$$-12 + 3x \leq 12,$$

$$3x \leq 24, \quad (\text{ we added } 12)$$

$$x \leq 8. \quad (\text{we divided by } 3)$$

To solve a quadratic inequality, we try to factor the quadratic. Failing that, we can complete the square and go on from there. This second method always works-

Example 2 Solve the inequality

$$x^2 - 4x + 3 > 0.$$

SOLUTION Factoring the quadratic, we obtain

$$(x - 1)(x - 3) > 0.$$

The product $(x - 1)(x - 3)$ is zero at 1 and 3. Mark these points on a number line

(Figure 1.3.1). The points 1 and 3 separate three intervals:

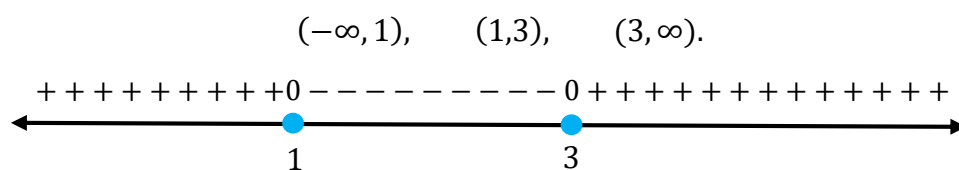


Figure 1.3.1

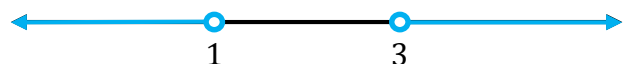
On each of these intervals the product $(x - 1)(x - 3)$ keeps a constant sign:

$$\text{on } (-\infty, 1) \text{ [to the left of 1] sign of } (x - 1)(x - 3) = (-)(-) = +;$$

$$\text{on } (1, 3) \text{ [between 1 and 3] sign of } (x - 1)(x - 3) = (+)(-) = -;$$

$$\text{on } (3, \infty) \text{ [to the right of 3] sign of } (x - 1)(x - 3) = (+)(+) = +.$$

The product $(x - 1)(x - 3)$ is positive on the open intervals $(-\infty, 1)$ and $(3, \infty)$. The solution set is the union $(-\infty, 1) \cup (3, \infty)$



Example 3 Solve the inequality

$$x^2 - 2x + 5 \leq 0$$

SOLUTION Not seeing immediately how to factor the quadratic, we use the method that always works: completing the square. Note that

$$x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x - 1)^2 + 4.$$

This tells us that

$$x^2 - 2x + 5 \geq 4 \quad \text{for all real } x$$

and thus there are no numbers that satisfy the inequality we are trying to solve. To put it in terms of sets, the solution set is the empty set $\emptyset$.

In practice we frequently come to expressions of the form

$$(x - a_1)^{k_1}(x - a_2)^{k_2} \dots (x - a_n)^{k_n}$$

$k_1, k_2 \dots k_n$ positive Integers, $a_1 < a_2 < \dots < a_n$. Such an expression is zero at $a_1, a_2 \dots a_n$. It is positive on those intervals where the number of negative factors is even and negative on those Intervals where the number of negative factors is odd.

Take, for instance,

$$(x + 2)(x - 1)(x - 3).$$

This product is zero at $-2, 1, 3$. It is

negative on $(-\infty, -2),$ (3 negative terms)

positive on $(-2, 1),$ (2 negative terms)

negative on $(1, 3),$ (1 negative term)

positive on $(3, \infty).$ (0 negative terms)

See Figure 1.3.2

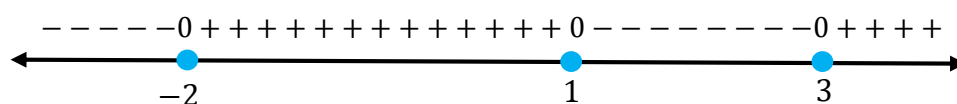


Figure 1.3.2

Example 4 Solve the inequality

$$(x + 3)^5(x - 1)(x - 4)^2 < 0.$$

SOLUTION We view $(x + 3)^5(x - 1)(x - 4)^2$ as the product of three factors:

$(x + 3)^5(x - 1)(x - 4)^2$. The product is zero at $-3, 1, 4$. These points separate the Intervals

$$(-\infty, -3), \quad (-3, 1), \quad (1, 4), \quad (4, \infty).$$

On each of these Intervals the product keeps a constant sign:

positive on $(-\infty, -3)$, (2 negative factors)

negative on $(-3, 1)$, (1 negative factor)

positive on $(1, 4)$, (0 negative factors)

positive on $(4, \infty)$. (0 negative factors)

See Figure 1.3.3.

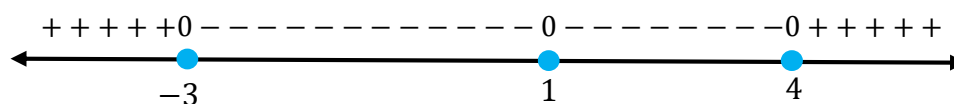


Figure 1.3.3

The solution set is the open Interval $(-3, 1)$.



Inequalities and Absolute Value:

As you know, for each real number a

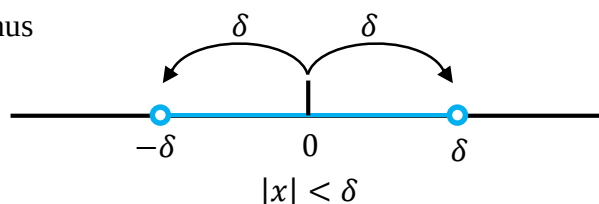
$$(1.3.1) \quad |a| = \begin{cases} a & \text{if } a \geq 0, \\ -a, & \text{if } a < 0, \end{cases} |a| = \max\{a, -a\}, |a| = \sqrt{a^2}.$$

We begin With the inequality

$$|x| < \delta$$

where δ is some positive number. To say that $|x| < \delta$ is to say that x lies within δ units of 0 or, equivalently, that x lies between $-\delta$ and δ . Thus

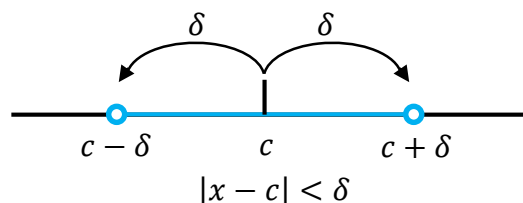
$$(1.3.2) \quad |x| < \delta \quad \text{iff} \quad -\delta < x < \delta.$$



The solution set is the open interval $(-\delta, \delta)$.

To say that $|x - c| < \delta$ is to say that x lies within δ units of c or, equivalently, that x lies between $c - \delta$ and $c + \delta$. Thus

$$(1.3.3) \quad |x - c| < \delta \quad \text{iff} \quad c - \delta < x < c + \delta.$$



The solution set is the open interval $(c - \delta, c + \delta)$

Somewhat more delicate is the inequality

$$0 < |x - c| < \delta.$$

Here we have $|x - c| < \delta$ with the additional requirement that $x \neq c$. Consequently,

$$(1.3.4) \quad 0 < |x - c| < \delta \quad \text{iff} \quad c - \delta < x < c \quad \text{or} \quad c < x < c + \delta.$$

The solution set is the union of two open intervals: $(c - \delta, c) \cup (c, c + \delta)$.

The following results are an immediate consequence of what we just showed.

$$|x| < \frac{1}{2} \text{ iff } -\frac{1}{2} < x < \frac{1}{2}; \quad [\text{solution set: } (-\frac{1}{2}, \frac{1}{2})]$$

$$|x - 5| < 1 \quad \text{iff} \quad 4 < x < 6; \quad [\text{solution set: } (4, 6)]$$

$$0 < |x - 5| < 1 \quad \text{iff} \quad 4 < x < 5 \quad \text{or} \quad 5 < x < 6; \quad [\text{solution set: } (4, 5) \cup (5, 6)]$$

Example 5 Solve the inequality

$$|x + 2| < 3.$$

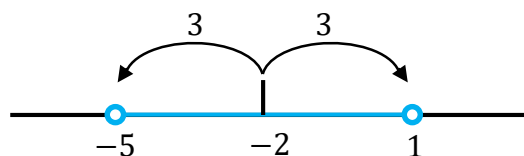
$$|x + 2| < 3.$$

$$|x + 2| = |x - (-2)|,$$

SOLUTION Once we recognize that $|x + 2| = |x - (-2)|$, we are in familiar territory.

$$|x - (-2)| < 3 \quad \text{iff} \quad -2 - 3 < x < -2 + 3 \quad \text{iff} \quad -5 < x < 1.$$

The solution set is the open interval $(-5, 1)$.



Example 6 Solve the inequality

$$|3x - 4| < 2.$$

SOLUTION Since

$$|3x - 4| = \left| 3 \left(x - \frac{4}{3} \right) \right| = |3| \left| x - \frac{4}{3} \right| = 3 \left| x - \frac{4}{3} \right|,$$

the inequality can be written

$$3 \left| x - \frac{4}{3} \right| < 2.$$

This gives

$$\left| x - \frac{4}{3} \right| < \frac{2}{3}, \quad \frac{4}{3} - \frac{2}{3} < x < \frac{4}{3} + \frac{2}{3}, \quad \frac{2}{3} < x < 2.$$

The solution set is the open interval $\left(\frac{2}{3}, 2 \right)$.

ALTERNATIVE SOLUTION There is usually more than one way to solve an inequality.

In this case, for example, we can write

$$|3x - 4| < 2$$

as

$$-2 < 3x - 4 < 2$$

and proceed from there. Adding 4 to the inequality, we get

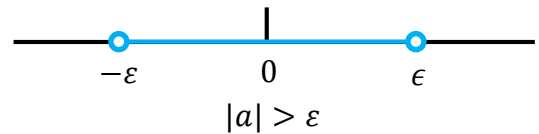
$$2 < 3x < 6.$$

Division by 3 gives the result we had before:

$$\frac{2}{3} < x < 2.$$

Let $\epsilon > 0$. If you think of $|a|$ as the distance between a and 0, then

$$(1.3.5) \quad |a| > \epsilon \quad \text{iff} \quad a > \epsilon \quad \text{or} \quad a < -\epsilon.$$



Example 7 Solve the inequality

$$|2x + 3| > 5.$$

SOLUTION In general

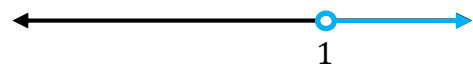
$$|a| > \epsilon \quad \text{iff} \quad a > \epsilon \quad \text{or} \quad a < -\epsilon.$$

So here

$$2x + 3 > 5 \quad \text{or} \quad 2x + 3 < -5.$$

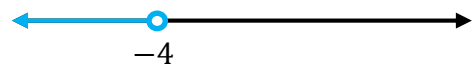
The first possibility gives $2x > 2$ and thus

$$x > 1.$$



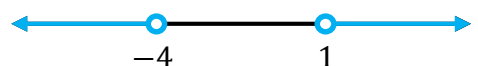
The second possibility gives $2x < -8$ and thus

$$x < -4$$



The total solution is therefore the union

$$(-\infty, -4) \cup (1, \infty).$$



We come now to one of the fundamental inequalities of calculus: for all real numbers

a and b ,

$$(1.3.6) \quad |a + b| \leq |a| + |b|.$$

This is called the triangle inequality in analogy With the geometric observation that "in any triangle the length of each side is less than or equal to the sum of the lengths of the Other sides."

Exercises

Exercises 1 – 20. Solve the inequality and mark the solution set on a number line.

1. $2 + 3x < 5.$

2. $\frac{1}{2}(2x + 3) < 6.$

3. $16x + 64 \leq 16$

4. $3x + 5 > \frac{1}{4}(x - 2).$

5. $\frac{1}{2}(1 + x) < \frac{1}{3}(1 - x).$

6. $3x - 2 \leq 1 + 6x.$

7. $x^2 - 1 < 0.$

8. $x^2 + 9x + 20 < 0.$

9. $x^2 - x - 6 \geq 0.$

10. $x^2 - 4x - 5 > 0.$

11. $2x^2 + x - 1 \leq 0.$

12. $3x^2 + 4x - 4 \geq 0.$

13. $x(x - 1)(x - 2) > 0.$

14. $x(2x - 1)(3x - 5) \leq 0.$

15. $x^3 - 2x^2 + x \geq 0.$

16. $x(2x - 1)(3x - 5) \leq 0.$

17. $x^3(x - 2)(x + 3)^2 < 0.$

18. $x^2 - 4x + 4 \leq 0.$

19. $x^2(x - 2)(x + 6) > 0.$

20. $x^2(x - 3)(x + 4)^2 > 0.$

Exercises 21 – 36. Solve the inequality and express the solution set as an interval or as the union of intervals.

21. $|x| < 2$.

22. $|x| \geq 1$.

23. $|x| > 3$.

24. $|x - 1| < 1$.

25. $|x - 2| < \frac{1}{2}$.

26. $|x - \frac{1}{2}| < 2$.

27. $0 < |x| < 1$.

28. $0 < |x| < \frac{1}{2}$.

29. $0 < |x - 2| < \frac{1}{2}$.

30. $0 < |x - \frac{1}{2}| < 2$.

31. $0 < |x - 3| < 8$.

32. $|3x - 5| < 3$.

33. $|2x + 1| < \frac{1}{4}$.

34. $|5x - 3| < \frac{1}{2}$.

35. $|2x + 5| > 3$.

36. $|3x + 1| > 5$.

Complex Numbers

Complex numbers are expressed in the form $a + ib$ or $a + bi$, where a and b are real numbers and i is a symbol for a new type of number whose square equals -1; that is, $i^2 = -1$.

We refer to a as the real part of the complex number and b as the imaginary part. The words “real” and “imaginary” have connotations that tend to place $\sqrt{-1}$ in a less favorable position in our minds than a real number such as $\sqrt{2}$. In fact, a good deal of imagination, or inventiveness, was required to construct the real number system, which forms the basis of calculus. We begin this chapter with a review of the successive stages of invented number systems.

The Complex Numbers

We have discussed these systems of numbers that form a hierarchy, with each system containing the previous system. Each system is richer than its predecessor in that it permits additional operations to be performed without going outside the system.

1. Using the integer system, we can solve all equations of the form

$$x + a = 0, \tag{1}$$

where a is an integer.

2. Using the rational numbers, we can solve all equations of the form

$$ax + b = 0, \tag{2}$$

provided that a and b are rational numbers and $a \neq 0$.

3. Using the real numbers, we can solve all of Equations (1) and (2) and, in addition, all quadratic equations

$$ax^2 + bx + c = 0, \tag{3}$$

provided that $a \neq 0$ and $b^2 - 4ac \geq 0$.

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{4}$$

gives the solution to Equation (3). When $b^2 - 4ac$ is negative, there are no real number solutions to the equation $ax^2 + bx + c = 0$. In particular, the simple quadratic equation $x^2 + 1 = 0$ cannot be solved using any of the three invented systems of numbers that we have discussed. Thus we are led to the fourth system of numbers, the set of complex numbers $a + ib$. The symbol i represents a new number whose square equals -1. We call a the real part and b the imaginary part of the complex number $a + ib$.

Real and Imaginary parts

For a complex number $z = a + ib$, we write

$$\operatorname{Re} z = a$$

and

$$\operatorname{Im} z = b$$

Sometimes it is convenient to write $a + bi$ instead of $a + ib$, both notations describe the same complex number. We define equality and addition for complex numbers in the following way.

Equality $a + ib = c + id$ if and only if $a = c$ and $b = d$.

Two complex numbers $a + ib$ and $c + id$ are equal if and only if their real parts are equal and their imaginary parts are equal.

Addition $(a + ib) + (c + id) = (a + c) + i(b + d)$

We sum the real parts and separately sum the imaginary parts.

To multiply two complex numbers, we multiply using the distributive rule and then simplify using $i^2 = -1$.

Multiplication $(a + ib) + (c + id)$

$$= ac + iad + ibc + i^2bd$$

$$= (ac - bd) + i(ad + bc) \quad i^2 = -1$$

EXAMPLE 1 Basic arithmetic operations with complex numbers

a. The set of all complex numbers $a + i0$, where the second number b is zero, has all of the properties of the set of real numbers. For example, addition and multiplication as complex numbers give

$$(a + i0) + (c + i0) = (a + c) + i0, \quad (a + i0)(c + i0) = ac + i0,$$

which are numbers of the same type with imaginary part zero. We usually just write a instead of $a + i0$, and in this sense the complex number system contains the real number system.

b. If we multiply a real number $a = a + i0$ by a complex number $c + id$, we get

$$a(c + id) = (a + i0)(c + id) = ac + iad.$$

c. The number $0 = 0 + i0$ plays the role of zero in the complex number system, and the complex number $1 = 1 + i0$ plays the role of unity, or one.

d. The complex number $i = 0 + i1$ which has real part zero and imaginary part one, has the property that its square is $i^2 = (0 + i1)^2 = (0 + i1)(0 + i1) = (-1) + i0 = -1$. So $x = i$ is a solution to the quadratic equation $x^2 + 1 = 0$. In the complex number system there are exactly two solutions to this equation, the other solution being $x = -i = 0 + i(-1)$.

e. We can divide any two complex numbers, as long as we do not divide by the number $0 = 0 + i0$. When $a + ib \neq 0$ (meaning that either $a \neq 0$ or $b \neq 0$ or both $a \neq 0$ and $b \neq 0$), we carry out division as follows:

$$\frac{c + id}{a + ib} = \frac{(c + id)(a - ib)}{(a + ib)(a - ib)} = \frac{(ac + bd) + i(ad - bc)}{a^2 + b^2} = \frac{ac + bd}{a^2 + b^2} + i \frac{ad - bc}{a^2 + b^2}$$

In this way we express the quotient in the standard form $x + iy$, where $x = (ac + bd)/(a^2 + b^2)$ and $y = (ad - bc)/(a^2 + b^2)$ are real numbers. Note that $a^2 + b^2 \neq 0$ since we stipulated that a and b cannot both be zero

f. The number $a - ib$ that was used in part (e) as the multiplier to clear the i from the denominator is called the complex conjugate of $a + ib$. If we denote the original complex number by $z = a + ib$, then it is customary to write $\bar{z}$ (read “ z bar”) to denote its complex conjugate:

$$z = a + ib, \quad \bar{z} = a - ib$$

Multiplying the numerator and denominator of a fraction $(c + id)/(a + ib)$ by the complex conjugate of the denominator will always replace the denominator by a real number.

EXAMPLE 2 Computations with complex numbers

$$(2 + 3i) + (6 - 2i) = (2 + 6) + (3 - 2)i = 8 + i$$

$$(2 + 3i) - (6 - 2i) = (2 - 6) + (3 - (-2))i = -4 + 5i$$

$$(2 + 3i)(6 - 2i) = (2)(6) + (2)(-2i) + (3i)(6) + (3i)(-2i)$$

$$= 12 - 4i + 18i - 6i^2 = 12 + 14i + 6 = 18 + 14i$$

$$\frac{2 + 3i}{6 - 2i} = \frac{(2 + 3i)(6 + 2i)}{(6 - 2i)(6 + 2i)} = \frac{12 + 4i + 18i + 6i^2}{36 + 12i - 12i - 4i^2} = \frac{6 + 22i}{40} = \frac{3}{20} + \frac{11}{20}i$$

The Complex Plane

We can use the usual xy -coordinate system to represent complex numbers as points or vectors in the plane. In this context we call it the *complex plane*. There are now two geometric representations of a complex number $z = x + iy$:

1. as a point $P(x, y)$ in the xy -plane,
2. as a vector $\overline{OP}$ from the origin to P .

In each representation, the x -axis is called the *real* axis and the y -axis is the *imaginary* axis, as in Figure 18.2. Geometric representations of complex numbers are sometimes called *Argand Diagrams*.

In polar coordinates we have

$$x = r \cos \theta, \quad y = r \sin \theta,$$

and

$$z = x + iy = r(\cos \theta + i \sin \theta). \quad (5)$$

The number 0, or $0 + i0$ is a special case, with $r = 0$ but no specified θ .

We define the absolute value (or magnitude) of a complex number $x + iy$ to be the length r of a vector $\overline{OP}$ from the origin to $P(x, y)$. We denote the absolute value by vertical bars: thus,

$$|x + iy| = \sqrt{x^2 + y^2}.$$

We always choose the polar coordinates r and θ so that r is nonnegative, and then

$$r = |x + iy|.$$

The polar angle θ is called the *argument* of z and is written $\theta = \arg z$. There is more than one choice for the argument. Any integer multiple of 2π may be added to θ to produce another appropriate angle. The following equation gives a useful formula connecting a complex number z , its conjugate $\bar{z}$, and its absolute value $|z|$:

$$z \cdot \bar{z} = |z|^2.$$

Euler's Formula

The identity

$$e^{i\theta} = \cos \theta + i \sin (\theta) \quad (6)$$

is called Euler's Formula. The origins of this formula will be seen in Section 18.6. We show a vector representing $e^{i\theta}$ in Figure 18.3. Using Equation (6), we can write Equation (5) as

$$z = re^{i\theta}.$$

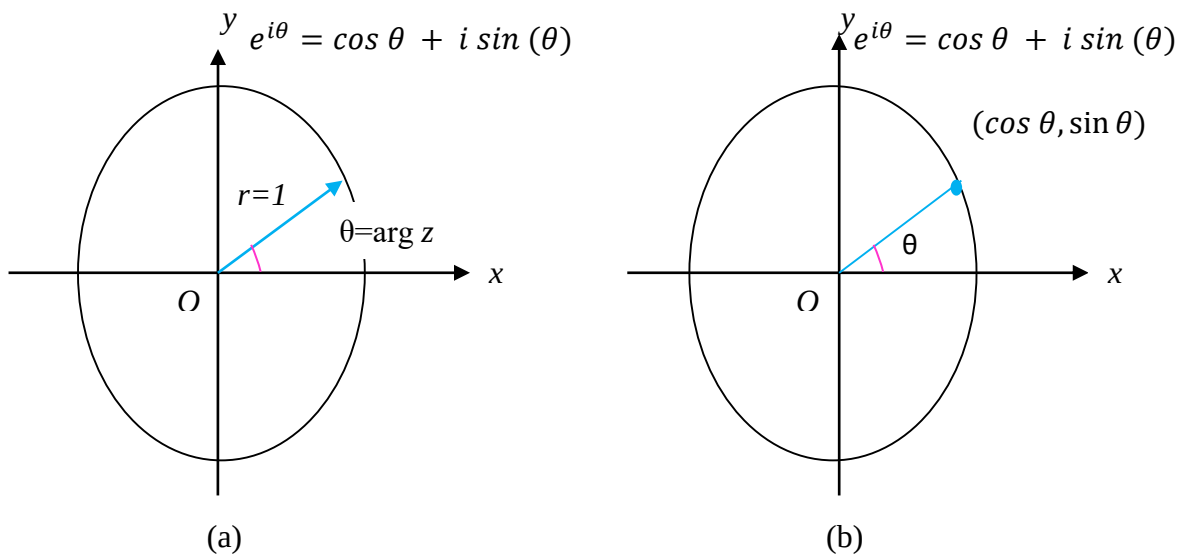


Figure 18.3 The complex number $z = e^{i\theta}$ as a vector and as a point

The notation $\exp(A)$ is also used for e^A , so we could write $z = r \exp(i\theta)$.

Euler's formula leads to the following rules for calculating products, quotients, powers, and roots of complex numbers in polar form.

Products

To multiply two complex numbers in polar form, we multiply their absolute values and add their arguments. To see why, let

$$z_1 = r_1 e^{i\theta_1} \text{ and } z_2 = r_2 e^{i\theta_2}, \quad (7)$$

so that

$$|z_1| = r_1, \arg z_1 = \theta_1; \quad |z_2| = r_2, \arg z_2 = \theta_2.$$

In the Cartesian $a + bi$ form we have

$$z_1 = r_1 \cos \theta_1 + i r_1 \sin \theta_1$$

and

$$z_2 = r_2 \cos \theta_2 + i r_2 \sin \theta_2$$

Their product is then

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 ([\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2] + i [\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2]) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \quad \text{Angle sum formulas} \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)}. \end{aligned}$$

Therefore

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

The absolute value of the product is $r_1 r_2$, and the argument of the product is the sum of the arguments of z_1 , and z_2 . We have shown that the product of two complex numbers is represented by a vector whose length is the product of the lengths of the two factors and whose angle with the x-axis is the sum of the corresponding angles of the factors. (Figure 18.4.) In particular, Equation (8) implies that multiplying a vector by $e^{i\theta}$ causes its vector representation to be rotated counterclockwise through an angle θ . Multiplication by i rotates 90° , by -1 rotates 180° , by $-i$ rotates 270° , and so on.

EXAMPLE 3 We compute the product of $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$. We first plot these complex numbers in Figure 18.5, and we can then read off their polar representations

$$\begin{aligned} z_1 \cdot z_2 &= \sqrt{2} \exp\left(\frac{i\pi}{4}\right) \cdot 2 \exp\left(-\frac{i\pi}{6}\right) \\ &= 2\sqrt{2} \exp\left(\frac{i\pi}{4} - \frac{i\pi}{6}\right) \\ &= 2\sqrt{2} \exp\left(\frac{i\pi}{12}\right) \\ &= 2\sqrt{2} \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right) \\ &\approx 2.73 + 0.73i. \end{aligned}$$

EXAMPLE 4 **Computing Reciprocals** Show that if $z = r e^{i\theta}$ and $r > 0$, then

$$\frac{1}{z} = \frac{1}{r} e^{-i\theta}.$$

Solution Applying the formula for products, Equation (8), we see that

$$(re^{i\theta})\left(\frac{1}{r}e^{-i\theta}\right) = 1e^{i0} = 1$$

Dividing both sides by $z = re^{i\theta}$ gives

$$\frac{1}{r}e^{-i\theta} = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

Quotients

We now look at how to divide $z_1 = r_1e^{i\theta_1}$ by $z_2 = r_2e^{i\theta_2}$ in polar form. We assume that $r_2 > 0$ so that we are not dividing by zero. To compute the quotient z_1/z_2 , we multiply z_1 by $1/z_2$. This gives

$$\frac{z_1}{z_2} = z_1 \left(\frac{1}{z_2}\right) = r_1e^{i\theta_1} \left(\frac{1}{r_2}e^{-i\theta_2}\right) = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}, \quad \text{Equation (8)}$$

so that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$$

That is, we divide lengths and subtract angles to compute the quotient of complex numbers.

EXAMPLE5 Let $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$ as in Example 3. Then

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2}e^{5\pi i/12} \approx 0.707 \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12}\right) \approx 0.183 + 0.683i.$$

Chapter 2:

2.1: linear equation in one variable

Def: A linear equation in one variable is an equation that can be written in the form

$$ax + b = 0$$

Where a and b are real numbers $a \neq 0$.

Ex (1): solve

(i) $5x - 4 = 11$

Solution

$$5x - 4 = 11$$

$$5x = 15$$

$$\longrightarrow x = 3 \quad (\text{divide each side by 5})$$

(ii) $3(2x - 4) = 7 - (x + 5)$

Solution:

$$3(2x - 4) = 7 - (x + 5)$$

$$6x - 12 = 7 - x - 5$$

$$6x - 12 = 2 - x$$

$$6x - x = 2 + 12$$

$$\longrightarrow 7x = 14$$

$$\longrightarrow x = 2$$

2.2: quadratic equation in one variable

Def: an equation that can be written in the form:

$$ax^2 + bx + c = 0 \quad , \text{ Where a, b and c are real numbers } a \neq 0 .$$

Ex (1): (solving quadratic equation by using factoring)

Solve:

(i) $x^2 - 6x - 16 = 0$

$$(x - 8)(x + 12) = 0$$

$$x - 8 = 0 \text{ or } x + 12 = 0$$

$$x = 8 \text{ or } x = -12 \quad * \text{ Solution set is } \{8, -12\}.$$

(ii) $x^2 - 8x = -12$

$$x^2 - 8x + 12 = 0$$

$$(x - 6)(x - 2) = 0$$

➡ $x - 6 = 0 \text{ or } x - 2 = 0$

➡ $x = 6 \text{ or } x = 2 \quad * \text{ Solution set is } \{6, 2\}$

Ex (2): (solving quadratic equation by using square root)

Solve:

(i) $x^2 = 17$

Solution:

$$x^2 = 17$$

Take $\sqrt{}$ to both sides

➡ $x = \pm\sqrt{17}$

(ii) $x^2 = -25$

Take $\sqrt{}$ to both sides

➡ $x = \pm\sqrt{-25}$

$$x = \pm 5i; \notin R$$

(iii) $(x - 4)^2 = 12$

Solution:

$$(x - 4)^2 = 12$$

Take $\sqrt{}$ to both sides

$$x - 4 = \pm\sqrt{12}$$

$$\longrightarrow x - 4 = \pm 2\sqrt{3}$$

$$\longrightarrow x = \pm 2\sqrt{3} + 4 \quad \text{*Solution set is } \{\pm 2\sqrt{3}, +4\}$$

Ex (3): (solving quadratic equation by using completing square)

Solve:

$$(i) \quad x^2 - 4x - 14 = 0$$

$$(ii) \quad \text{add } \left(\frac{-4}{2}\right)^2 \text{ to both sides}$$

$$x^2 - 4x + 4 = 14 + 4$$

$$(iii) \quad (x - 2)^2 = 18$$

$$(iv) \quad \text{Take } \sqrt{} \text{ to both sides}$$

$$x - 2 = \pm\sqrt{18}$$

$$\longrightarrow x - 2 = \pm 3\sqrt{2}$$

$$\longrightarrow x = \pm 3\sqrt{2} + 2 \quad \text{*Solution set is } \{\pm 3\sqrt{2}, +2\}$$

** Quadratic Formula:

The solution of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$

Are given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ex (4):

Solve the following quadratic equation by using quadratic formula:

$$(i) \quad x^2 - 4x = -2$$

Solution:

$$x^2 - 4x + 2 = 0$$

$$a = 1, b = -4, c = 2$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{(2)(1)}$$

$$x = \frac{4 \pm \sqrt{16 - 8}}{2}$$

$$x = \frac{4 \pm \sqrt{8}}{2}$$

$$x = \frac{4 \pm 2\sqrt{2}}{2}$$

$$x = \frac{4}{2} + \frac{2\sqrt{2}}{2}$$

$$x = 2 \pm \sqrt{2}$$

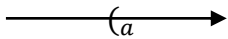
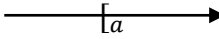
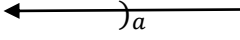
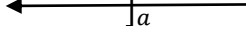
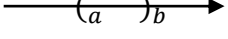
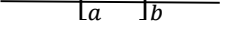
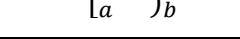
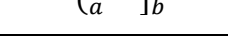
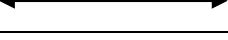
*Solution set is $\{2 \pm \sqrt{2}\}$

2.3: linear inequation in one variable

Def: A linear inequation in one variable is an inequation that can be written in the form:

$$ax + b > 0$$

Where a and b are real numbers $a \neq 0$.

Inequality	Interval notation	Type	Graph
$x > a$	(a, ∞)	Open	
$x \geq a$	$[a, \infty)$	closed	
$x < a$	$(-\infty, a)$	Open	
$x \leq a$	$(-\infty, a]$	closed	
$a < x < b$	(a, b)	Open	
$a \leq x \leq b$	$[a, b]$	closed	
$a \leq x < b$	$[a, b)$	Half Open	
$a < x \leq b$	$(a, b]$	Half Open	
Real number	$(-\infty, \infty)$	Open	

Ex: solve and graph the following inequalities:

(i) $3x - 2 > 1$

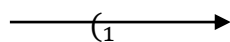
Solution :

$$3x - 2 > 1$$

$$3x > 3$$

$$x > 1$$

*Solution set is $(1, \infty)$



(ii) $-3x + 5 > -7$

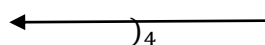
Solution:

$$-3x + 5 > -7$$

$$-3x > -12 \quad (\text{divide both side by } -3)$$

$$x < 4$$

*Solution set is $(-\infty, 4)$

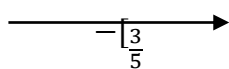


$$(iii) \quad 4 - 3x \leq 7 + 2x$$

$$-5x \leq 3$$

$$x \geq \frac{-3}{5}$$

*Solution set is $[\frac{-3}{5}, \infty)$



$$(iv) \quad -2 < 3x + 5 < 20$$

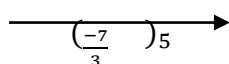
Solution:

$$-2 < 3x + 5 < 20$$

$$-7 < 3x < 15$$

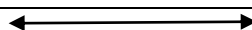
$$-\frac{7}{3} < x < 5$$

*Solution set is $(-\frac{7}{3}, 5)$



2.4: Absolute value equations and inequalities

Absolute value equations or inequalities	Equivalent form	graph	Solution Set
$ x = k$	$x = k \text{ or } x = -k$		$\{-k, k\}$
$ x < k$	$-k < x < k$		$(-k, k)$
$ x \leq k$	$-k \leq x \leq k$		$[-k, k]$



$ x > k$	$x < -k \text{ or } x > k$	$)_{-k} \quad (k$	$(-\infty, -k) \cup (k, \infty)$
$ x \geq k$	$x \leq -k \text{ or } x \geq k$	$\longleftarrow]_{-k} \quad [k \longrightarrow$	$(-\infty, -k] \cup [k, \infty)$

Ex (1): solve each equation

(i) $|5-3x|=12$

(ii) $|4x + 3| - 2 = -1$

Solution:

(i) $|5 - 3x| = 12$

$$5 - 3x = 12 \quad \text{or} \quad 5 - 3x = -12$$

$$-3x = 7 \quad \text{or} \quad -3x = -17$$

$$x = \frac{-7}{3} \quad \text{or} \quad x = \frac{17}{3}$$

The solution set is $\{\frac{-7}{3}, \frac{17}{3}\}$

(ii) $|4x + 3| - 2 = -1$

Solution:

$$|4x + 3| - 2 = -1$$

$$|4x + 3| = 1$$

$$4x + 3 = 1 \quad \text{or} \quad 4x + 3 = -1$$

$$4x = -2 \quad \text{or} \quad 4x = -4$$

$$x = \frac{-1}{2} \quad \text{or} \quad x = -1$$

The solution set is $\{\frac{-1}{2}, -1\}$

Ex (2): solve each inequality

(i) $|2x + 1| < 7$

(ii) $|2 - 7x| - 1 > 5$

(i) $-7 < 2x + 1 < 7$

$$-8 < 2x < 6$$

$$-4 < x < 3$$

The solution set is $(-4, 3)$

$$(ii) \quad |2 - 7x| - 1 > 5$$

$$|2 - 7x| > 6$$

$$2 - 7x > 6 \quad \text{or} \quad 2 - 7x < -6$$

$$-7x > 4 \quad \text{or} \quad -7x < -8$$

$$x < \frac{4}{-7} \quad \text{or} \quad x > \frac{8}{7}$$

The solution set is $(-\infty, \frac{-4}{7}) \cup (\frac{8}{7}, \infty)$

Chapter 3:

3.1: Functions

Def: A function is relation between two sets, where each element in the first set, called (the domain) corresponds to exactly one element in the second set, called (the range)

Ex (1): Determine whether the following relations are functions.

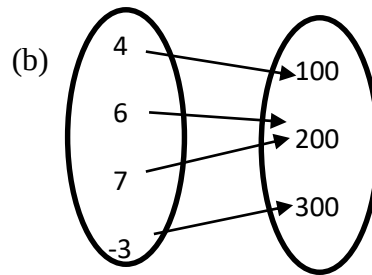
- (i) $F = \{(1,2), (-2,4), (3,4)\}$ (ii) $G = \{(1,1), (1,2), (1,3), (2,3)\}$

Solution:

- (i) F is function (ii) G is not function.

Ex (2): Find the Domain and range of each relation.

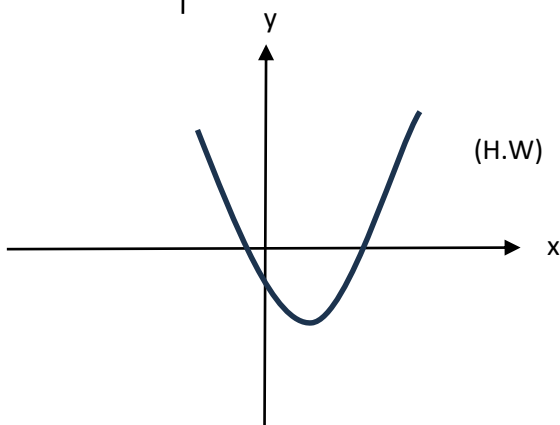
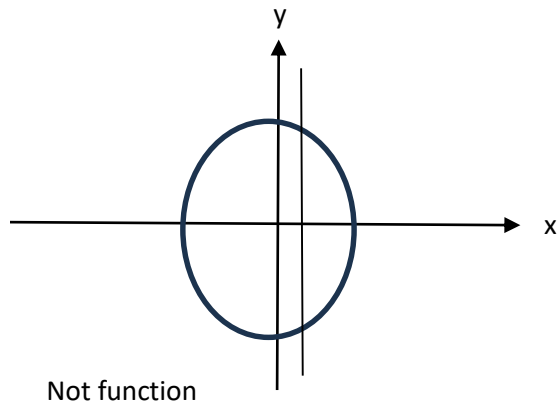
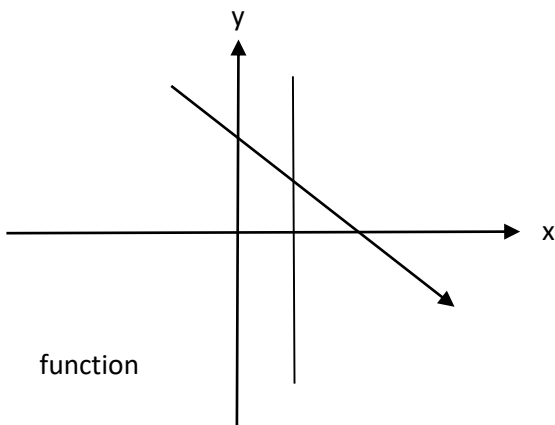
- (a) $\{(3, -1), (4,2), (4,5), (6,8)\}$



Solution:

- (a) Domain = $\{3,4,6\}$, Range = $\{-1,2,5,8\}$

Ex (3): use the vertical line test to determine whether each graph is a function



Ex (4): let $f(x) = -x^2 + 5x - 3$, and $g(x) = 2x + 3$, Find and simplify:

(i) $f(2)$, (ii) $f(z)$, (iii) $g(a + 1)$

Sol:

(i) $f(2) = -(2)^2 + 5(2) - 3 = -4 + 10 - 3 = 3$

(ii) $f(z) = -z^2 + 5z - 3$

(iii) $g(a + 1) = 2(a+1) + 3$
 $= 2a + 2 + 3$
 $= 2a + 5$

Ex (5): let $f = \{(-3,5), (0,3), (3,1), (6, -1)\}$, Find $f(3)$.

Sol: $f(3) = 1$

Ex (6): Find the domain of each function:

(i) $y = x + 4$ (ii) $y = \sqrt{2x - 1}$ (iii) $y = \frac{5}{x-1}$

Solution:

(i) the Domain = $\mathbb{R} = (-\infty, +\infty)$

(ii) the Domain $\longrightarrow 2x - 1 \geq 0$

$$2x \geq 1$$

$$x \geq \frac{1}{2}$$

The Domain = $[\frac{1}{2}, \infty]$

(iii) Domain $\longrightarrow x - 1 = 0$

$$x = 1$$

Domain = $\{X: X \neq 1\}$ or $\mathbb{R} - \{1\}$

3.2: (i) Graphing Linear Functions

Def: A function $y = f(x)$ is a linear function if $y = ax + b$

Where a and b are real numbers $a \neq 0$.

Intercepts.

(i) X – intercepts, put $y = 0$, then the point is $(?,0)$

(ii) Y - intercepts, put $X = 0$, then the point is $(0,?)$

Ex(1): Graph $y = -2x + 6$

Solution :

(i) Find the X – intercept ($y=0$)

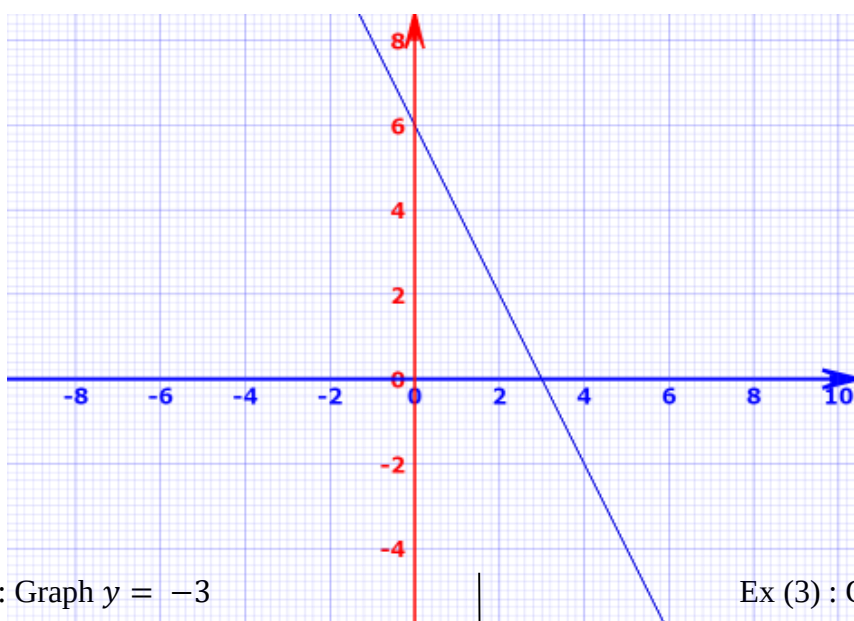
$$\longrightarrow 0 = -2x + 6 \longrightarrow -2x = -6 \longrightarrow x = 3$$

X – intercept is $(3,0)$

(ii) Find the Y – intercept ($x=0$)

$$\longrightarrow y = -2(0) + 6 \longrightarrow y = 6$$

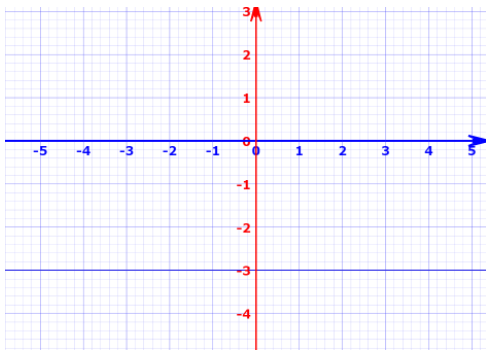
Y – intercept Is $(0,6)$



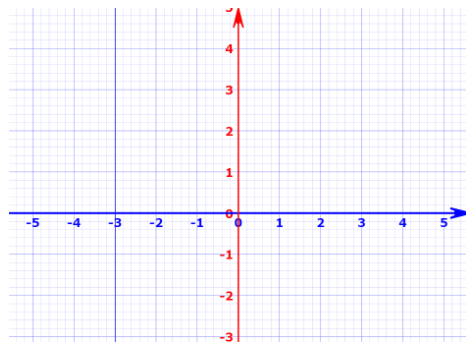
Ex (2) : Graph $y = -3$

Ex (3) : Graph $x = -3$

Solution :



solution :



(ii) Slope of Lines

Def : the slope “m” of the line through the points (x_1, y_1) and (x_2, y_2) is given by the following :

$$m = \frac{y_2 - y_1}{x_2 - x_1}, (x_1 \neq x_2) \longrightarrow (1)$$

Ex(1) : find the slope of line through the given points .

(i) $(-4, 8), (2, -3)$

(ii) $(2, 7), (2, -4)$

(iii) $(5, -3), (-2, -3)$

Solutions :

(i) $m = \frac{-3-8}{2+4} = -\frac{11}{6}$

(ii) $m = \frac{-4-7}{2-2} = -\frac{11}{0}$, undefined

(iii) $m = \frac{-3+3}{-2-5} = -\frac{0}{7} = 0$

(iii) Finding the slope from equation

If $AX + BY = C$ is standard form of equation of line, then:

$$\boxed{\text{Slope (m)} = -\frac{A}{B}} \longrightarrow (2)$$

Ex (2) : find the slope of line $4x + 3y = 12$.

Solution:

$$A = 4, B = 3$$

$$m = -\frac{4}{3}$$

3.3: 1- Equations of lines

(i) Point – slope form

The point – slope form of the equation of line with slope (m) passing through the point (x1, y1) is:

$$\boxed{y - y_1 = m(x - x_1)} \longrightarrow (1)$$

Ex(1): Write the equation of line that passes through (-4,1) with slope =-3

Solutions: given $x_1 = -4$, $y_1 = 1$, $m = -3$

The equation is:

$$y - y_1 = m(x - x_1)$$

$$\longrightarrow y - 1 = -3(x + 4)$$

$$\longrightarrow y = -3x - 12 + 1$$

$$\longrightarrow y = -3x - 11 \quad \underline{\text{Or}} \quad y + 3x + 11 = 0$$

Ex(2): write the equation of line that passes through (-3,2) and (2, -4) (H.W)

(ii) Slope - Intercept Form

The slope - intercept Form of the equation of the LiNe with slope (m) and y.

intercept (o, b) is:

$$Y = mX + b \longrightarrow (2)$$

EX (3): Find the slope and Y - intercept of the Line of equation $4x + 5 + y = -10$

solutions:

We rewrite the equation $4x + 5y = -10$ as the Form $y = mx + b$

$$\longrightarrow 5y = -4x - 10$$

Divide both sides by 5

$$\longrightarrow y = -\frac{4}{5}x - 2$$

By comparing with Equation (2)

$$\longrightarrow m = -\frac{4}{5}, b = -2$$

2- Parallel Lines

Two Lines are parallel if and only if they have the same slope. $\longrightarrow (m_1 = m_2) \longrightarrow$
(i)

Where :

$m_2 \equiv$ slope of Line parallel to m_1

Ex(4): Find the slope of Line parallel to $2x + 5y = 4$.

Solution :

First Find m_1 , $A=2$, $B=5$

$$\longrightarrow m_1 = -\frac{A}{B} = -\frac{2}{5}$$

$$\longrightarrow m_2 = -\frac{2}{5} \quad (\text{Parallel to } m_1)$$

3- Perpendicular Lines

Two Lines are Perpendicular if and only if their slope product is -1.

→ $M_1 \cdot M_2 = -1$

Where:

$$m_2 = \frac{-1}{m_1} \equiv \text{slope of line Perpendicular to } m_1$$

Ex (5) : Find the slope of Line perpendicular to the Line $2x + 5y = 4$

Solution: First find m_1 , $A=2$, $B=5$

→ $m_1 = \frac{-A}{B} = \frac{-2}{5}$

→ $m_2 = \frac{5}{2}$ (perpendicular to m_1)

3.5: Even and Odd Functions

Def:

(i) A function $f(x)$ is an even function if:

$$f(-x) = f(x)$$

(ii) A function $f(x)$ is an odd function if:

$$f(-x) = -f(x)$$

Ex: Determine whether each function is even, odd, or neither.

(a) $f(x) = 8x^4 - 3x^2$, (b) $f(x) = 6x^3 - 9x$, (c) $f(x) = 3x^2 + 5x$

Solution : (a) $f(x) = 8x^4 - 3x^2$

$$\begin{aligned} f(-x) &= 8(-x)^4 - 3(-x)^2 \\ &= 8x^4 - 3x^2 = f(x) \end{aligned}$$

$f(x) = 8x^4 - 3x^2$ is even .

(b): $f(x) = 6x^3 - 9x$

$$f(-x) = 6(-x)^3 - 9(-x)$$

$$= -6x^3 + 9x = -f(x)$$

$$f(x) = 6x^3 - 9x \text{ is odd}$$

$$(c): f(x) = 3x^2 + 5x$$

$$f(-x) = 3(-x)^2 + 5(x)$$

$$= 3x^2 - 5x \neq f(x) \quad = -(-3x^2 + 5x) \neq -f(x)$$

$$f(x) = 3x^2 + 5x \text{ is neither.}$$

3.6:

(i) Operations on functions:

Given two functions f and g , then for all Values of x which both $f(x)$ and $g(x)$ are defined as follows:

$$(a) \quad (f \pm g)(x) = f(x) \pm g(x)$$

$$(b) \quad (f \cdot g)(x) = f(x) \cdot g(x)$$

$$(c) \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad , \quad g(x) \neq 0$$

Ex (1): let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$, Find each of the following:

$$(i) \quad (f + g)(x) \quad (ii) \quad (f - g)(x) \quad (iii) \quad (f \cdot g)(x)$$

Solution:

$$(i) \quad (f + g)(x) = f(x) + g(x) \\ = (x^2 + 1) + (3x + 5) = x^2 + 3x + 6$$

$$(ii) \quad (f - g)(x) = f(x) - g(x) \\ = (x^2 + 1) - (3x + 5) = x^2 - 3x - 4$$

$$(iii) \quad (f \cdot g)(x) = f(x) \cdot g(x) \\ = (x^2 + 1)(3x + 5) \\ = 3x^3 + 5x^2 + 5$$

Ex(2): let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$, Find each of the following:

$$(i) \quad (f + g)(1) \quad (ii) \quad (f - g)(-3) \quad (iii) \quad (f \cdot g)(5) \quad (iv) \quad \left(\frac{f}{g}\right)(0)$$

Solution:

$$(i) \quad (f + g)(1) , \quad f(1) = 1^2 + 1 = 2 \quad g(1) = 3(1) + 5 = 8 \\ = 2+8 \\ =10$$

$$(ii) \quad (H.W)$$

$$(iii) \quad (f \cdot g)(5) = f(5) \cdot g(5) = f(5) = 5^2 + 1 = 26 , g(5) = 3(5) + 5 = 20 \\ = (26)(20) \\ = 520$$

$$(iv) \quad (H.W)$$

(ii) Composition on functions:

If f and g are functions, then the Composition. Of f and g is defined by:

$$(f \circ g)(x) = f(g(x))$$

Ex (1): given that $f(x) = \sqrt{x}$ and $g(x) = 4x + 2$, find the following:

$$(i) \quad (f \circ g)(x) \qquad (ii) \quad (g \circ f)(x)$$

Solution:

$$\begin{aligned} (i) \quad (f \circ g)(x) &= f(g(x)). \\ &= f(4x + 2) \\ &= \sqrt{4x + 2} \end{aligned}$$

$$\begin{aligned} (ii) \quad (g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt{x}) \\ &= 4\sqrt{x} + 2 \end{aligned}$$

Ex (2): given that $f(x) = 2x - 1$ and $g(x) = \frac{4}{x-1}$, find:

$$(i) \quad (f \circ g)(2) \qquad (ii) \quad (g \circ f)(-3)$$

Solution:

$$\begin{aligned} (i) \quad (f \circ g)(2) &= f(g(2)) \\ &= f\left(\frac{4}{2-1}\right) \\ &= f(4) \\ &= 2(4) - 1 = 7 \end{aligned}$$

$$\begin{aligned} (ii) \quad (g \circ f)(-3) &= g(f(-3)) \\ &= g(2(-3) - 1) \\ &= g(-7) \\ &= \frac{4}{-7-1} = \frac{4}{-8} \\ &= -\frac{1}{2} \end{aligned}$$

Chapitre 5

5-1 (i) one –to– one Functions

A function $f(x)$ is a one –to– one function, if for elements a and b ,

$$f(a) = f(b) \Rightarrow a = b$$

EX(1) Determine whether each function is one – to –one :

(i) $f(x) = 4x + 12$

(ii) $f(x) = \sqrt{25 - x^2}$

Solution

$$\Rightarrow f(a) = f(b)$$

$$\Rightarrow -4a + 12 = -4b + 12$$

$$\Rightarrow -4a = -4b$$

$$\Rightarrow a = b$$

$$\Rightarrow f(x) = 4x + 12 \quad \text{is one to one}$$

(i) Let $a=3$, $b=-3$

$$\Rightarrow f(3) = \sqrt{25 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$$

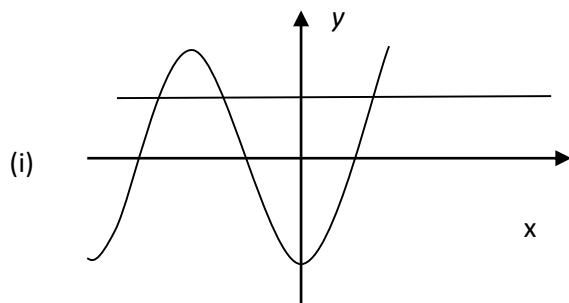
$$\Rightarrow f(-3) = \sqrt{25 - (-3)^2} = \sqrt{25 - 9} = \sqrt{16} = 4$$

$$\Rightarrow f(3) = f(-3)$$

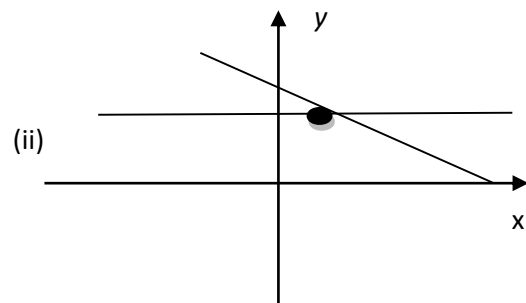
$$\Rightarrow f(x) = \sqrt{25 - x^2} \text{ is not one to one}$$

Ex (2) : using the horizontal line test

Determine whether each graph is one – to– one fuction :



Not one to one function



One –to –one function

(ii) a Inverse function

Let f be one- to- one function , then f^{-1} is the inverse of f

Ex(1) find the inverse of each function that is one- to one

(i) $F = \{(3,1), (0,2), (2,3), (4,0)\}$

(ii) $G = \{(-2,1), (-1,0), (0,1), (1,2), (2,2)\}$

Solution :

(i) F is one –to- one function

$\Rightarrow f^{-1} = \{(1,3), (1,2), (3,2), (0,4)\}$

(ii) G is not one – to – one function

$\Rightarrow G$ Does not have inverse

(iii) Finding the inverse of equation $y = f(x)$

For a one – to one function f defined by an equation $y = f(x)$ find the inverse as follow :

Step 1 : inerchange x and y

Step 2 : solve for y ,

Step 3 : Replace y by $f(x)^{-1}$.

Ex : Find the inverse of each equation

(i) $y = 2x + 5$ (ii) $y = (x - 2)^2$

Solution

(i) $\Rightarrow x = 2y + 5$

$\Rightarrow x - 5 = 2y$

$\Rightarrow y = \frac{x-5}{2}$

$\Rightarrow f(x)^{-1} = \frac{x-5}{2}$

(ii) $x = (y - 2)^3$

$\Rightarrow \sqrt[3]{x} = y - 2$

$\Rightarrow \sqrt[3]{x} + 2 = y$

$f(x)^{-1} = \sqrt[3]{x} + 2$

5.2 Exponential Functions

Definition if $a > 0$ and $a \neq 1$ then $f(x) = a^x$

Defines the exponential function within base a

Ex : If $f(x) = 2^x$, Find each of the following :

$$(i) \quad f(-1) \quad (ii) \quad f(3) \quad (iii) \quad f\left(\frac{5}{2}\right)$$

Solution :

$$(i) \quad f(-1) = 2^{-1} = \frac{1}{2}$$

$$(ii) \quad f(3) = 2^3 = 8$$

$$(iii) \quad f\left(\frac{5}{2}\right) = 2^{5/2} = (2^5)^{1/2} = \sqrt{32} = \sqrt{16 * 2} = 4\sqrt{2}$$

One to one property :

$$\text{If } a^{x_1} = a^{x_2} \quad \Rightarrow, \text{ then } x_1 = x_2$$

Ex (1) = solve :

$$\left(\frac{1}{3}\right)^x = 81$$

Solution

$$\Rightarrow \left(\frac{1}{3}\right)^x = 3^4$$

$$\Rightarrow 3^{-x} = 3^4$$

$$\Rightarrow -x = 4$$

$$\Rightarrow x = -4$$

Ex (2) = solve :

$$2^{x+4} = 8^{x-6}$$

Solution :

$$2^{x+4} = (2^3)^{x-6}$$

$$\Rightarrow 2^{x+4} = 2^{3x-18}$$

$$\Rightarrow x + 4 = 3x - 18$$

$$\Rightarrow -2x = -22$$

$$\Rightarrow x = 11$$

5.3 (i) Logarithmic functions

Def if $a > 0$, $a \neq 1$ and $x > 0$ then $f(x) = \log_a x$ defines the logarithmic function with base a

Note :

$$\text{If } y = \log_a x \Leftrightarrow x = a^y$$

Ex(1) : (Convert the logarithmic form to exponential form)

Express each equation in equivalent exponential form

$$(i) \quad \log_2 8 = 3 \Rightarrow 2^3 = 8$$

$$(ii) \quad \log_{1/2} 16 = -4 \Rightarrow (1/2)^{-4} = 16$$

$$(iii) \quad \log_3 1/81 = -4 \Rightarrow 3^{-4} = 1/81 \quad 3^{-4} = 1/81$$

$$(iv) \quad \log_5 5 = 1 \Rightarrow 5^1 = 5$$

$$(v) \quad \log_{3/4} 1 = 0 \Rightarrow (3/4)^0 = 1 \quad (3/4)^0 = 1$$

Ex (2) Solve each equation :

$$(i) \quad \log_x \frac{8}{27} = 3 \quad (ii) \quad \log_4 x = \frac{5}{2} \quad (iii) \quad \log_{49} 3\sqrt{7} = x$$

Solution :

$$(i) \quad x^3 = \frac{8}{27} \text{ take } \sqrt[3]{\quad} \text{ to both sides}$$

$$\Rightarrow x = \left(\frac{8}{27}\right)^{1/3}$$

$$\Rightarrow x = \left(\left(\frac{2}{3}\right)\right)^{3^{1/3}} = \frac{2}{3}$$

$$(ii) \quad (4)_{5/2}^{5/2} = x$$

$$\Rightarrow (2^2)^{5/2} = x$$

$$\Rightarrow x = 2^5 = 32$$

$$(iii) \quad 49^x = \sqrt[3]{7}$$

$$\Rightarrow 49^x = 7^{1/3}$$

$$\Rightarrow (7^2)^x = 7^{1/3}$$

$$\Rightarrow 7^{2x} = 7^{1/3}$$

$$\Rightarrow 2x = \frac{1}{3}$$

$$\Rightarrow x = \frac{1}{6}$$

(ii) Domain of logarithmic function

$$f(x)$$

$$\text{If } y = \log_a x, \text{ then domain } \Rightarrow g(x) > 0$$

Ex2 Find the domain of each function

$$(i) \quad f(x) = \log_2(x - 1) \quad (ii) \quad \log_3(5 - 2x)$$

Solution :

$$(i) \quad \text{Domain } \Rightarrow x - 1 > 0$$

$$\Rightarrow x > 1$$

$$\Rightarrow (1, \infty)$$

$$(ii) \quad \text{Domain } 5 - 2x > 0$$

$$\Rightarrow -2x > -5$$

$$\Rightarrow x < \frac{5}{2}$$

$$\Rightarrow (-\infty, \frac{5}{2})$$

(iii) properties of logarithms

$$(i) \quad \log_a xy = \log_a x + \log_a y \text{ (Product property)}$$

$$(ii) \quad \log_a \frac{x}{y} = \log_a x - \log_a y \text{ (Quotient property)}$$

$$(iii) \quad \log_a x^{\sqrt[n]{n}} = \sqrt[n]{n} \log_a x \text{ (Power property)}$$

$$(iv) \quad \log_a 1 = 0$$

$$(v) \quad \log_a a = 1$$

Ex(1) Rewrite each expression

$$(i) \quad \log_6 7 * 9 = \log_6 7 + \log_6 9$$

$$(ii) \quad \log_9 \left(\frac{15}{7} \right) = \log_9 15 - \log_9 7$$

$$(iii) \quad \log_5 \sqrt{8} \log_5 8^{\frac{1}{2}} = \frac{1}{2} \log_5 8$$

$$(iv) \quad \log_a \frac{mnz}{p^2 t^4} = \log_a mnz - \log_a (p^2 t^4)$$

$$= \log_a m + \log_a n + \log_a z - (\log_a p^2 \log_a t^4)$$

$$= \log_a m \log_a n \log_a z - 2 \log_a p - 4 \log_a t$$

$$(v) \quad \log_a \sqrt[3]{m^2} = \log_a m^{2/3} = \frac{2}{3} \log_a m$$

$$\begin{aligned}
 \text{(vi)} \quad \log_b \sqrt[n]{\frac{x^3 y^5}{z^m}} &= \log_b \left(\frac{x^3 y^5}{z^m} \right)^{\frac{1}{n}} \\
 &= \frac{1}{n} \log_b \left(\frac{x^3 y^5}{z^m} \right) \\
 &= \frac{1}{n} [\log_b x^3 + \log_b y^5 + \log_b z^m] \\
 &= \frac{3}{n} \log_b x + \frac{5}{n} \log_b y - \frac{m}{n} \log_b z
 \end{aligned}$$

Ex(2) write each expressions as a single logarithm

$$\begin{aligned}
 \text{(i)} \quad &\log_3(x+2) + \log_3 x - \log_3 2 \\
 \text{(ii)} \quad &2 \log_a m - 3 \log_a n \\
 \text{(iii)} \quad &\frac{1}{2} \log_b m + \frac{3}{2} \log_b 2n - \log_b m^2 n
 \end{aligned}$$

Solution :

$$\begin{aligned}
 \text{(i)} \quad \log_3(x+2) + \log_3 x - \log_3 2 &= \log_3 x(x+2) - \log_3 2 = \log_3 \frac{x(x+2)}{2} \\
 &= \log_2 \frac{(x^2+2x)}{2}
 \end{aligned}$$

$$\text{(ii)} \quad 2 \log_a m - 3 \log_a n = \log_a m^2 - \log_a n^3 = \log_a \frac{m^2}{n^3}$$

Introduction to geometry:

Chapter2:

The circle

I- I-Standard equation of a circle:

1) Definition:

A circle is the locus of a point that moves in such a way that its distance from a fixed point is always a constant.

The fixed point of a circle is called the center, and the constant distance is called the radius of the circle.

2) Equation of the circle (standard form):

Let $C(a, b)$ be the center of the circle and $r \geq 0$ its radius, then the **equation of the circle** is given by:

$$(x - a)^2 + (y - b)^2 = r^2.$$

FIG

Example1:

If the center of a circle is $(3, -2)$ and the radius is 4, what is the equation of the circle.

$$(x - 3)^2 + (y + 2)^2 = 4^2$$

$$\Leftrightarrow x^2 - 6x + 9 + y^2 + 4y + 4 - 16 = 0$$

$$\Leftrightarrow x^2 + y^2 - 6x + 4y - 3 = 0.$$

Example2:

If the center of a circle is $(2, -1)$ and the radius is 3, what is the equation of the circle.

$$(x - 2)^2 + (y + 1)^2 = 3^2$$

$$\Leftrightarrow x^2 - 4x + 4 + y^2 + 2y + 1 - 9 = 0$$

$$\Leftrightarrow x^2 + y^2 - 4x + 2y - 4 = 0.$$

3) General Equation of a circle:

Let

$$(x - a)^2 + (y - b)^2 = r^2$$

be the standard equation of a circle, by developing, we obtain

$$\begin{aligned} (x - a)^2 + (y - b)^2 = r^2 &\Leftrightarrow x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0 \\ &\Leftrightarrow x^2 + y^2 + Dx + Ey + F = 0 \end{aligned}$$

Where $D = -2a, E = -2b, F = a^2 + b^2 - r^2$,

Which gives us **the general form**:

$$x^2 + y^2 + Dx + Ey + F = 0.$$

Reciprocally, every second-order equation of the form

$$x^2 + y^2 + Dx + Ey + F = 0$$

represents a circle.

To prove this result, we proceed

$$\begin{aligned} x^2 + y^2 + Dx + Ey + F = 0 &\Leftrightarrow x^2 + Dx + y^2 + Ey + F = 0 \\ &\Leftrightarrow x^2 + Dx + \left(\frac{D}{2}\right)^2 + y^2 + Ey + \left(\frac{E}{2}\right)^2 + F - \left(\frac{D}{2}\right)^2 - \left(\frac{E}{2}\right)^2 = 0 \\ &\Leftrightarrow \left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \left(\frac{D}{2}\right)^2 + \left(\frac{E}{2}\right)^2 - F \\ &\Leftrightarrow \left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2 - 4F}{4} \end{aligned}$$

which represents a circle with center $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ and radius

$$r = \sqrt{\frac{D^2 + E^2 - 4F}{4}} \quad (\text{if } \frac{D^2 + E^2 - 4F}{4} \geq 0)$$

Remark: if $\frac{D^2 + E^2 - 4F}{4} < 0$, the set of points $P(x, y)$ such that

$$x^2 + y^2 + Dx + Ey + F = 0$$

is empty.

Example3:

Change the equation

$$4x^2 + 4y^2 - 12x + 4y - 26 = 0$$

FIG

to the standard form.

Solution:

$$\begin{aligned}
 4x^2 + 4y^2 - 12x + 4y - 26 = 0 & \Leftrightarrow x^2 + y^2 - 3x + y - \frac{26}{4} = 0 \\
 & \Leftrightarrow x^2 - 3x + y^2 + y - \frac{26}{4} = 0 \\
 & \Leftrightarrow x^2 - 2\frac{3}{2}x + \left(\frac{3}{2}\right)^2 + y^2 + 2\frac{1}{2}y + \left(\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \frac{26}{4} = 0 \\
 & \Leftrightarrow \left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 - \frac{9}{4} - \frac{1}{4} - \frac{26}{4} = 0 \\
 & \Leftrightarrow \left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 - \frac{36}{4} = 0 \\
 & \Leftrightarrow \left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{36}{4} = 9 = 3^2 \\
 & \Leftrightarrow \left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 3^2
 \end{aligned}$$

Which is the standard equation of the circle with center $\left(\frac{3}{2}, -\frac{1}{2}\right)$ and radius 3.

Example4:

Find the general equation of the circle that has the center

$(1, -3)$ and pass through the point $(4, 5)$.

Solution:

The standard equation of the circle is

$$(x - 1)^2 + (y + 3)^2 = r^2,$$

(4,5) belong to the circle, then $(4 - 1)^2 + (5 + 3)^2 = r^2$

$$\begin{aligned} &\Leftrightarrow 73 = r^2 \\ &\Leftrightarrow r = \sqrt{73} \end{aligned}$$

So, we have

$$(x - 1)^2 + (y + 3)^2 = 73.$$

$$\begin{aligned} (x - 1)^2 + (y + 3)^2 = 73 &\Leftrightarrow x^2 - 2x + 1 + y^2 + 6y + 9 = 73 \\ &\Leftrightarrow x^2 + y^2 - 2x + 6y - 63 = 0 \end{aligned}$$

is the general equation of the circle.

Example5:

Find the general equation of the circle that pass through the

points $(2, -1)$, $(0,2)$ and $(1,1)$.

Solution:

The general equation of the circle is of the form:

$$C: x^2 + y^2 + Dx + Ey + F = 0$$

$$(2, -1) \in C \Leftrightarrow 4 + 1 + 2D - E + F = 0$$

$$(0,2) \in C \Leftrightarrow 4 + 2E + F = 0$$

$$(1,1) \in C \Leftrightarrow 1 + 1 + D + E + F = 0$$

$$\begin{cases} 2D - E + F = -5 \\ 2E + F = -4 \\ D + E + F = -2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2D - E + F = -5 \\ F = -4 - 2E \\ D = -2 - E - F \pm 2 - E - (-4 - 2E) = E + 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2(E + 2) - E - 4 - 2E = -5 \Leftrightarrow E = 5 \\ F = -4 - 2 \times 5 = -14 \\ D = E + 2 = 7 \end{cases}$$

Then, we get

$$C: x^2 + y^2 + 7x + 5y - 14 = 0$$

4) Conditions that determine a circle:

Example6:

- a) Find the equation of the circle that pass through the points $(1,2)$, $(-2,1)$ and $(2,-3)$.
- b) Find the center and the radius of the circle.

Solution:

- a) The general equation of the circle is of the form:

$$C: x^2 + y^2 + Dx + Ey + F = 0$$

$$(1,2) \in C \leftrightarrow 1 + 4 + D + 2E + F = 0$$

$$(-2,1) \in C \leftrightarrow 4 + 1 - 2D + E + F = 0$$

$$(2,-3) \in C \leftrightarrow 4 + 9 + 2D - 3E + F = 0$$

$$\begin{aligned} \begin{cases} D + 2E + F = -5 \\ -2D + E + F = -5 \\ 2D - 3E + F = -13 \end{cases} &\leftrightarrow \begin{cases} D + 2E + F = -5 \\ R_2 \leftarrow R_2 + 2R_1: 5E + 3F = -15 \\ R_3 \leftarrow R_3 - 2R_1: -7E - F = -3 \end{cases} \\ &\leftrightarrow \begin{cases} D + 2E + F = -5 \\ 5E + 3(-7E + 3) = -15 \\ F = -7E + 3 \end{cases} \\ &\leftrightarrow \begin{cases} D + 2E + F = -5 \\ -16E + 9 = -15 \\ F = -7E + 3 \end{cases} \\ &\leftrightarrow \begin{cases} D + 2E + F = -5 \\ E = \frac{24}{16} = \frac{3}{2} \\ F = -7\frac{3}{2} + 3 = -\frac{15}{2} \end{cases} \\ &\leftrightarrow \begin{cases} D + 2\frac{3}{2} - \frac{15}{2} = -5 \leftrightarrow D = -\frac{1}{2} \\ E = \frac{3}{2} \\ F = -\frac{15}{2} \end{cases} \end{aligned}$$

Then, we get

$$C: x^2 + y^2 - \frac{1}{2}x + \frac{3}{2}y - \frac{15}{2} = 0$$

- b) Let 's change the general equation to the standard form:

$$\begin{aligned}
x^2 + y^2 - \frac{1}{2}x + \frac{3}{2}y - \frac{15}{2} &= 0 \Leftrightarrow x^2 - \frac{1}{2}x + y^2 + \frac{3}{2}y - \frac{15}{2} = 0 \\
&\Leftrightarrow x^2 - 2\frac{1}{4}x + \frac{1}{16} - \frac{1}{16} + y^2 + 2\frac{3}{4}y + \frac{9}{16} - \frac{9}{16} - \frac{15}{2} = 0 \\
&\quad ((x-a)^2 = x^2 - 2ax + a^2) \\
&\Leftrightarrow \left(x - \frac{1}{4}\right)^2 - \frac{1}{16} + \left(y - \frac{3}{4}\right)^2 - \frac{9}{16} - \frac{15}{2} = 0 \\
&\Leftrightarrow \left(x - \frac{1}{4}\right)^2 + \left(y - \frac{3}{4}\right)^2 = \frac{1}{16} + \frac{9}{16} + \frac{15}{2} = \frac{130}{16} \\
&\Leftrightarrow \left(x - \frac{1}{4}\right)^2 + \left(y - \frac{3}{4}\right)^2 = \left(\frac{\sqrt{130}}{4}\right)^2
\end{aligned}$$

Then, the radius is $r = \frac{\sqrt{130}}{4}$ and the center is $\left(\frac{1}{4}, \frac{3}{4}\right)$

Example7:

Find the equation of the circle that pass through the points $(1,2)$ and $(-2,1)$ and having its center on the line

$$x - y + 2 = 0.$$

Solution:

The standard form of the equation of such circle is

$C: (x - a)^2 + (y - b)^2 = r^2$, where (a, b) is the center and r is the radius.

$$(1,2) \in C \Leftrightarrow (1 - a)^2 + (2 - b)^2 = r^2$$

$$(-2,1) \in C \Leftrightarrow (-2 - a)^2 + (1 - b)^2 = r^2$$

$$(a, b) \in D: x - y + 2 = 0 \Leftrightarrow a - b + 2 = 0$$

$$\begin{aligned}
&\Leftrightarrow \begin{cases} (1-a)^2 + (2-b)^2 = r^2 \\ (-2-a)^2 + (1-b)^2 = r^2 \\ a-b+2=0 \end{cases} \\
&\Leftrightarrow \begin{cases} (1-a)^2 + (2-(a+2))^2 = r^2 \\ (-2-a)^2 + (1-(a+2))^2 = r^2 \\ b=a+2 \end{cases} \\
&\Leftrightarrow \begin{cases} (1-a)^2 + a^2 = r^2 \\ (-2-a)^2 + (a+1)^2 = r^2 \\ b=a+2 \end{cases} \\
&\Leftrightarrow \begin{cases} 2a^2 - 2a + 1 = r^2 \\ 2a^2 + 6a + 5 = r^2 \\ b=a+2 \end{cases} \\
&\Leftrightarrow \begin{cases} 2a^2 - 2a = r^2 \\ 8a + 4 = 0 \\ b=a+2 \end{cases} \\
&\Leftrightarrow \begin{cases} 2 \times \frac{1}{4} - 2 \times \left(-\frac{1}{2}\right) + 1 = \frac{5}{2} = r^2 \\ a = -\frac{1}{2} \\ b = \frac{3}{2} \end{cases}
\end{aligned}$$

Then, we get

$$C: \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{5}{2}$$

5) Families of circles:

5-1 Family of intersecting circles: $k \neq -1$.

Consider the equations of two intersecting circles, C_1 and C_2 :

$$\begin{aligned}
C_1: x^2 + y^2 + D_1x + E_1y + F_1 &= 0 \\
C_2: x^2 + y^2 + D_2x + E_2y + F_2 &= 0
\end{aligned}$$

Then, if we take k as a parameter, we can take the sum of the given

equations to get the equation of the **family of circles**:

$$x^2 + y^2 + D_1x + E_1y + F_1 + k(x^2 + y^2 + D_2x + E_2y + F_2) = 0$$

By collecting terms, we can obtain the form:

$$(1+k)x^2 + (1+k)y^2 + (D_1 + kD_2)x + (E_1 + kE_2)y + (F_1 + kF_2) = 0,$$

Which we recognize as the equation of a circle for every value of k except $k = -1$:

$$x^2 + y^2 + \frac{D_1 + kD_2}{1 + k}x + \frac{E_1 + kE_2}{1 + k}y + \frac{F_1 + kF_2}{1 + k} = 0.$$

Example8:

a) Find the equation of the circle that crosses the Y-axis at $(0, -4)$

and passes through the intersections of circles C_1 and C_2 :

$$C_1: x^2 + y^2 - 2x - 2y - 2 = 0$$

$$C_2: x^2 + y^2 + 4x - 6y + 4 = 0.$$

b) Draw the three circles.

Example9:

Find the equations of the circles that passes through the

intersections of the circles C_1 and C_2 :

And the point $(2,1)$.

5-2) Radical axis of two circles: $k = -1$.

When $k = -1$, the particular equation

$$C_1 + kC_2 = 0$$

becomes

And assumes a **linear form**:

$$(D_1 - D_2)x + (E_1 - E_2)y + (F_1 - F_2) = 0.$$

Such equation represents a straight line called the **Radical axis of two circles**.

Remarks:

1) If two circles have no real points in common, they may still have a

radical axis.

2) If two circles intercept at real points, the radical axis is called the common chord.

3) If two circles are tangents, it is called the common tangent.

Example10:

a) Find the radical axis of the circles C_1 and C_2 :

$$\begin{aligned}C_1: x^2 + y^2 + 4x + 2y - 1 &= 0 \\C_2: x^2 + y^2 - 8x - 4y + 12 &= 0.\end{aligned}$$

b) Draw the circles and the radical axis.

Standard Equation of an Ellipse

The standard form of the equation of an ellipse with center (h, k) and major and minor axes of lengths $2a$ and $2b$, where $a > b$, is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

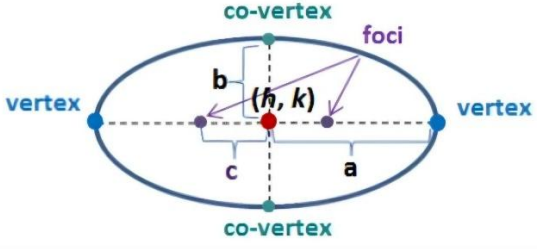
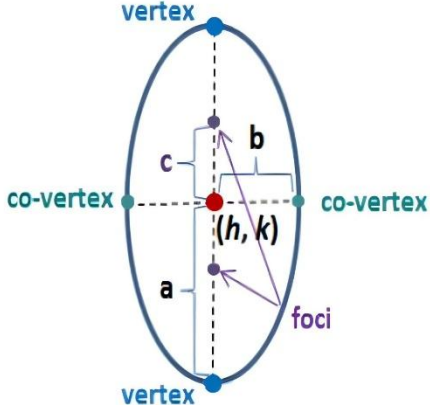
Major axis is horizontal

or

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Major axis is vertical

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$.

Horizontal Ellipse	Vertical Ellipse
At $(0,0)$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ General: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ $a^2 - b^2 = c^2$ Center: (h, k) Foci: $(h \pm c, k)$ Vertices: $(h \pm a, k)$ Co-Vertices: $(h \pm b, k)$ 	At $(0,0)$: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ General: $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ $a^2 - b^2 = c^2$ Center: (h, k) Foci: $(h, k \pm c)$ Vertices: $(h, k \pm a)$ Co-Vertices: $(h \pm b, k)$ 

find the vertices and foci of the ellipse.

$$1. \frac{x^2}{16} + \frac{y^2}{7} = 1$$

Center: (0,0)

$$\begin{aligned} \text{Foci: } c &= \sqrt{16-7} \\ &= \sqrt{9} \\ &= \pm 3 \end{aligned}$$

$$(\pm c, k) = (\pm 3, 0)$$

$$\begin{aligned} \text{Vertices: } \pm\sqrt{a^2} &= \pm 4 \\ \pm\sqrt{b^2} &= \pm 2.6 \\ (\pm 4, 0) \quad (0, \pm 2.6) \end{aligned}$$

$$2. \frac{x^2}{25} + \frac{y^2}{21} = 1$$

Center: (0,0)

$$\begin{aligned} \text{Foci: } c &= \sqrt{25-21} \\ &= \sqrt{4} \\ &= \pm 2 \end{aligned}$$

$$(h, \pm c) = (0, \pm 2)$$

$$\begin{aligned} \text{Vertices: } \pm\sqrt{21} &= \pm 4.6 \\ \pm\sqrt{25} &= \pm 5 \\ (\pm 4.6, 0) \quad (0, \pm 5) \end{aligned}$$

$$9. \frac{(y-2)^2}{16} + \frac{(x+3)^2}{4} = 1$$

Center: (-3,2) = (h,k)

$$a^2 = 16, b^2 = 4, c = \sqrt{12} = 2\sqrt{3}$$

$$\text{Foci} = (h, k + c) = (-3, 2 + 2\sqrt{3})$$

$$\text{and } = (h, k - c) = (-3, 2 - 2\sqrt{3})$$

$$\text{Vertices} = (h, k + a) = (-3, 6)$$

$$\text{and } = (h, k - a) = (-3, -2)$$

$$\text{Co - Verices} = (h + b, k) = (-1, 1)$$

$$\text{and } = (h - b, k) = (-5, 1)$$

$$10. \frac{(x-1)^2}{11} + (y+2) = 1$$

Standard Equation of a Parabola:

The **standard form** of the equation of a parabola with vertex (h, k) and directrix

$y = k - p$ is

$$(x - h)^2 = 4p(y - k) \quad \textbf{Vertical axis}$$

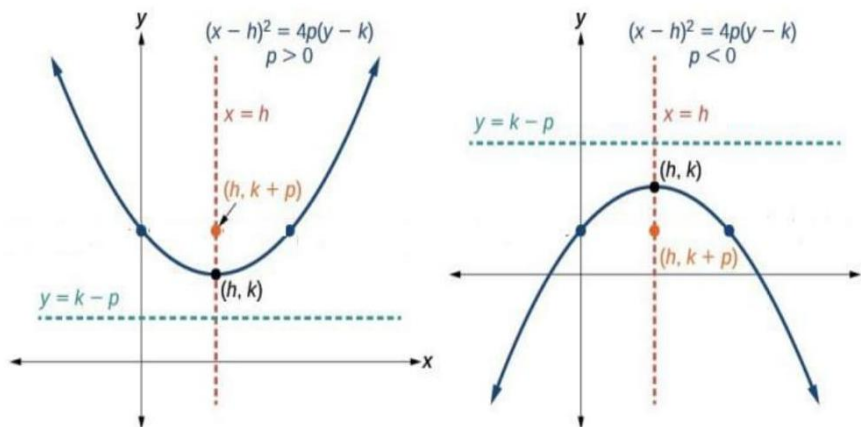
For directrix $x = h - p$, the equation is

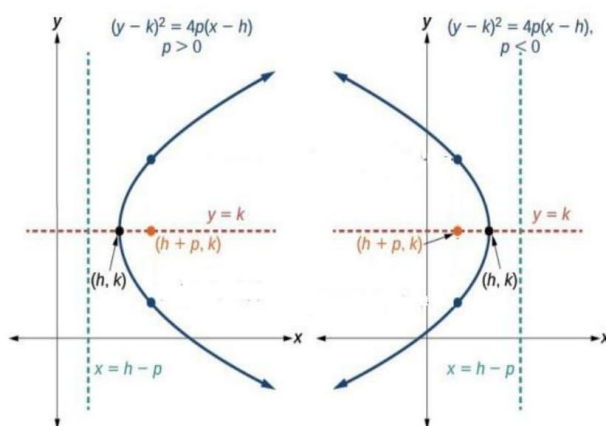
$$(y - k)^2 = 4p(x - h) \quad \textbf{Horizontal axis}$$

The focus lies on the axis p units (directed distance) from the vertex. The coordinates of the focus are as follows.

$$(h, k + p) \quad \textbf{Vertical axis}$$

$$(h + p, k) \quad \textbf{Horizontal axis}$$





Examples:

Find the vertex, focus, directrix, of the parabola:

1. $x^2 = 6y$

$$4P = 6$$

$$P = \frac{6}{4}$$

$$\Rightarrow P = \frac{3}{2}$$

$$V: (h, k) = (0, 0)$$

$$F: (h, k + p)$$

$$= (0, \frac{3}{2})$$

$$D: y = k - p$$

$$= 0 - \frac{3}{2} = -\frac{3}{2}$$

2. $y^2 = -8x$

$$4P = -8$$

$$P = \frac{-8}{4}$$

$$\Rightarrow P = -2$$

$$V: (h, k) = (0, 0)$$

$$F: (h + p, k)$$

$$= (-2, 0)$$

$$D: x = h - P$$

$$= 0 - (-2) = 2$$

3. $(y - 2)^2 =$

$$4(x + 3)$$

$$4P = 4$$

$$\Rightarrow P = 1$$

$$V: (-3, 2)$$

$$F: (h + p, k)$$

4. $(x + 4)^2 =$

$$-6(y + 1)$$

$$4P = -6$$

$$\Rightarrow P = -\frac{3}{2}$$

$$V: (-4, -1)$$

$$F: (h, k + p)$$

$$(-3+1,2)$$

$$(-2,2)$$

$$(-4,-1+\frac{-3}{2})$$

$$\left(-4,\frac{-5}{2}\right)$$

In exercices 1-6 find the vertex, Focus , directrix, the parabola.

$$1. \quad x^2 = 6$$

$$\frac{1}{6}x^2 = y$$

$$p = \frac{1}{4(\frac{1}{6})}$$

$$= \frac{1}{\frac{4}{6}}$$

$$= \frac{3}{2}$$

$$V : (h, K) = (0, 0)$$

$$F : (h, K + p) = (0, \frac{3}{2})$$

$$D : y = K - p = 0 - \frac{3}{2} = -\frac{3}{2}$$

$$3. \quad (y - 2)^2 = 4(x + 3)$$

$$\frac{1}{4}(y - 2)^2 = (x + 3)$$

$$p = \frac{1}{4(\frac{1}{4})} = 1$$

$$V : (-3, 2)$$

$$F : (h + p, K)$$

$$(-3 + 1, 2)$$

$$(-2, 2)$$

$$D : x = h - p$$

$$= -3 - 1$$

$$= -4$$

$$5. \quad 3x^2 = -4y$$

$$2. \quad y^2 = -8x$$

$$-\frac{1}{8}y^2 = x$$

$$p = \frac{1}{4(-\frac{1}{8})}$$

$$p = \frac{1}{-\frac{4}{8}}$$

$$= -2$$

$$V : (h, K) = (0, 0)$$

$$F : (h, K + p) = (-2, 0)$$

$$D : x = h - p = 0 - (-2) = 2$$

$$4. \quad (x + 4)^2 = -6(y + 1)$$

$$-\frac{1}{6}(x + 4)^2 = (y + 1)$$

$$p = \frac{1}{4(-\frac{1}{6})} = -\frac{3}{2}$$

$$V : (-4, -1)$$

$$F : (h, K + p)$$

$$(-4, -1 + (-\frac{3}{2}))$$

$$(-4, -\frac{5}{2})$$

$$y = K - p$$

$$= -1 - (-\frac{3}{2})$$

$$= \frac{1}{2}$$

$$6. \quad 5y^2 = 16x$$

$$-\frac{3}{4}x^2 = y$$

$$p = \frac{1}{4(\frac{-3}{4})} = -\frac{1}{3} =$$

$$V : (h, K) = (0, 0)$$

$$F : (h, K + p)$$

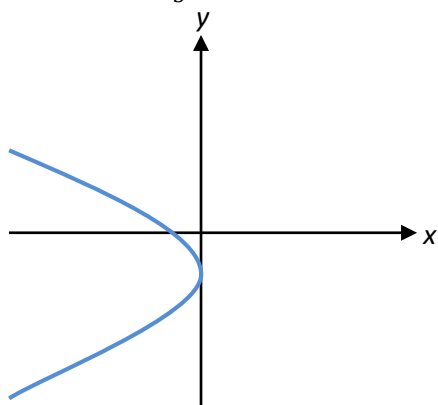
$$(0, 0 + (-\frac{1}{3}))$$

$$(0, -\frac{1}{3})$$

$$D : y = K - p$$

$$= 0 - (-\frac{1}{3})$$

$$= \frac{1}{3}$$



(a)

$$\frac{5}{16}y^2 = x$$

$$p = \frac{1}{4(\frac{5}{16})} = \frac{1}{\frac{5}{6}} = \frac{4}{5}$$

$$V : (h, K) = (0, 0)$$

$$F : (h + p, K)$$

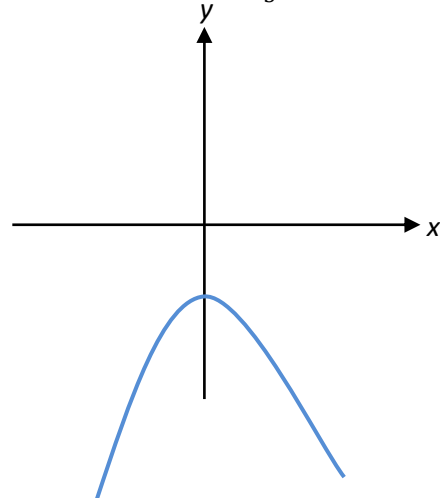
$$(0 + \frac{4}{5}, 0)$$

$$(\frac{4}{5}, 0)$$

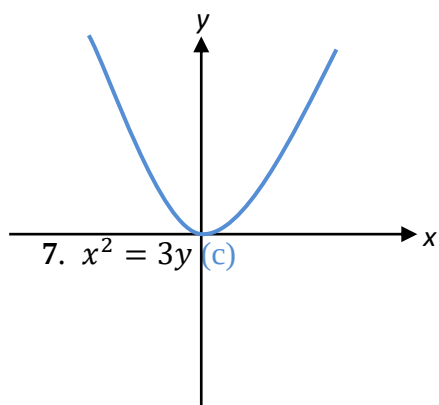
$$D : x = h - p$$

$$= 0 - \frac{4}{5}$$

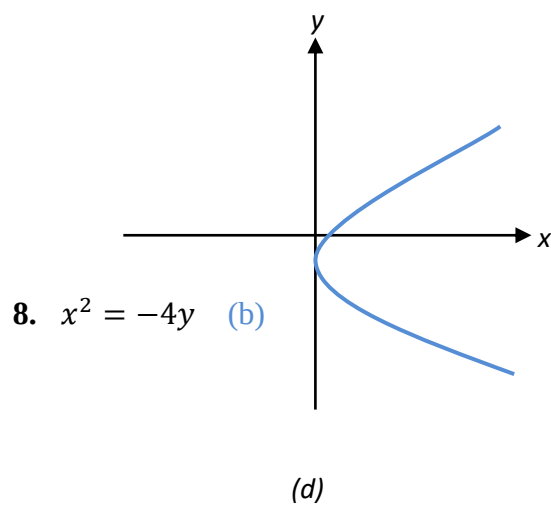
$$= -\frac{4}{5}$$



(b)



(c)



(d)

9. $y^2 = -5x$ (a)

10. $y^2 = 10x$ (d)

In Exercises 31-36, sketch the graph of the parabola by hand

31. $y^2 = -4x$

$$-\frac{1}{4}y^2 = x$$

$$p = \frac{1}{4(-\frac{1}{4})} = -1$$

$$V : (0,0)$$

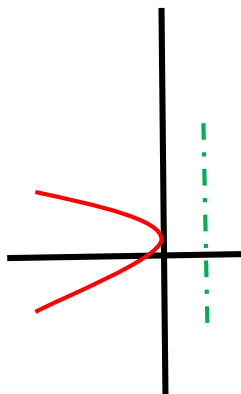
$$F : (h + p, K)$$

$$(-1,0)$$

$$D : x = h - p$$

$$= 0 - (-1)$$

$$= 1$$



32. $x^2 = 8y$

$$\frac{1}{8}x^2 = y$$

$$p = \frac{1}{4(\frac{1}{8})} = 2$$

$$V = (0,0)$$

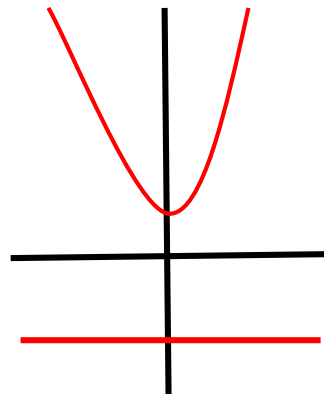
$$F = (h, K + p)$$

$$(0,2)$$

$$D = K - p$$

$$= 0 - 2$$

$$= -2$$



33. $(x + 4)^2 = 12(y + 1)$

$$-\frac{1}{12}(x + 4)^2 = (y + 1)$$

$$p = \frac{1}{4(-\frac{1}{12})} = -3$$

$$V = (-4,1)$$

$$F : (h, K + p)$$

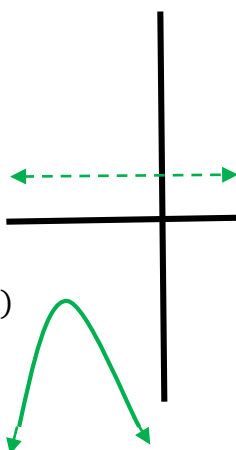
$$(-4, -1 + -3)$$

$$(-4, -4)$$

$$D : y = K - p$$

$$= -1 - -3$$

$$= +2$$



34. $(y + 2)^2 = 16(x + 3)$

$$-\frac{1}{16}(y + 2)^2 = (x + 3)$$

$$p = \frac{1}{4(-\frac{1}{16})} = -4$$

$$V : (-3, -2)$$

$$F : (h + p, K)$$

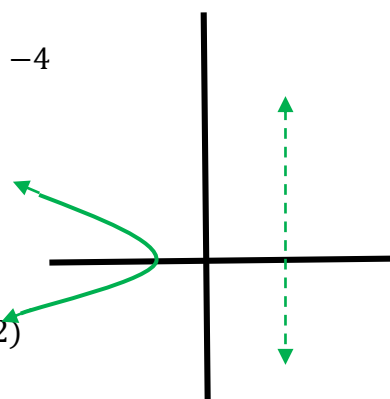
$$(-3 + 4, -2)$$

$$(-7, -2)$$

$$D : x = h - p$$

$$= -3 - -4$$

$$= 1$$



$$35. (y - 1)^2 = 8(x + 3)$$

$$\frac{1}{8}(y - 1)^2 = (x + 3)$$

$$p = \frac{1}{4(\frac{1}{8})} = 2$$

$$V : (-3, 1)$$

$$F : (h + p, K)$$

$$(-3 + 2, 1)$$

$$(-1, 1)$$

$$D : x = h - p$$

$$= -3 - 2$$

$$= -5$$

$$36. (x - 5)^2 = 20(y + 2)$$

$$\frac{1}{20}(x - 5)^2 = (y + 2)$$

$$p = \frac{1}{4(\frac{1}{20})} = 5$$

$$V : (5, -2)$$

$$F : (h, K + p)$$

$$(5, -2 + 5)$$

$$(5, 3)$$

$$D : y = K - p$$

$$= -2 - 5$$

$$= -7$$

In exercises 1-6 find the vertices and foci of the ellipse.

$$1. \frac{x^2}{16} + \frac{y^2}{7} = 1$$

$$\text{Center} : (0, 0)$$

$$\text{Foci} : C = \sqrt{16 - 7}$$

$$= \sqrt{9}$$

$$= \pm 3$$

$$(\pm C, K) = (\pm 3, 0)$$

$$\text{Vertices} : \pm\sqrt{a^2} = \pm 4$$

$$\pm\sqrt{b^2} = \pm 2.6$$

$$(\pm 4, 0) \quad (0, \pm 2.6)$$

$$3. \frac{y^2}{36} + \frac{x^2}{27} = 1$$

$$2. \frac{y^2}{25} + \frac{x^2}{21} = 1$$

$$\text{Center} : (0, 0)$$

$$\text{Foci} : C = \sqrt{25 - 21}$$

$$= \sqrt{4}$$

$$= \pm 2$$

$$(h, \pm C) = (0, \pm 2)$$

$$\text{Vertices} : \pm\sqrt{21} = \pm 4.6$$

$$\pm\sqrt{25} = \pm 5$$

$$(\pm 4.6, 0) \quad (0, \pm 5)$$

$$4. \frac{x^2}{11} + \frac{y^2}{7} = 1$$

Center : (0,0)

$$\text{Foci : } C = \sqrt{36 - 27}$$

$$= \sqrt{9}$$

$$= \pm 3$$

$$(h, \pm C) = (0, \pm 3)$$

$$\text{Vertices : } \pm\sqrt{a^2} = \sqrt{27} = \pm 5.2$$

$$\pm\sqrt{b^2} = \sqrt{36} = \pm 6$$

$$(\pm 5.2, 0) \quad (0, \pm 6)$$

$$5. \frac{3x^2}{12} + \frac{4y^2}{12} = \frac{12}{12}$$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Center : (0,0)

$$\text{Foci : } C = \sqrt{4 - 3}$$

$$= \sqrt{1}$$

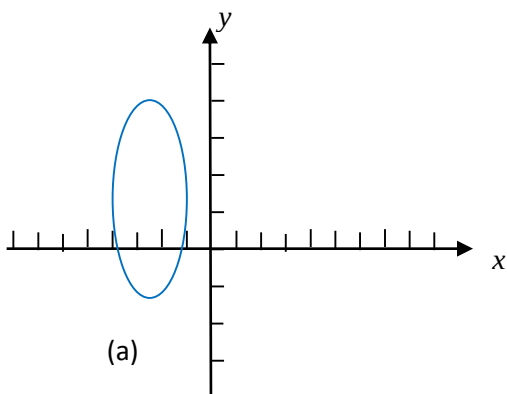
$$= \pm 1$$

$$(\pm 1, 0)$$

$$\text{Vertices : } \sqrt{a^2} = \sqrt{4} = \pm 2$$

$$\sqrt{b^2} = \sqrt{3} = \pm 1.7$$

$$(\pm 2, 0) \quad (0, \pm 1.7)$$



Center : (0,0)

$$\text{Foci : } C = \sqrt{11 - 7}$$

$$= \sqrt{4}$$

$$= \pm 2$$

$$(\pm C, K) = (\pm 2, 0)$$

$$\text{Vertices : } \sqrt{a^2} = \sqrt{11} = \pm 3.6$$

$$\sqrt{b^2} = \sqrt{7} = \pm 2.6$$

$$(\pm 3.6, 0) \quad (0, \pm 2.6)$$

$$6. \frac{9x^2}{36} + \frac{4y^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Center : (0,0)

$$\text{Foci : } C = \sqrt{9 - 4}$$

$$= \sqrt{5}$$

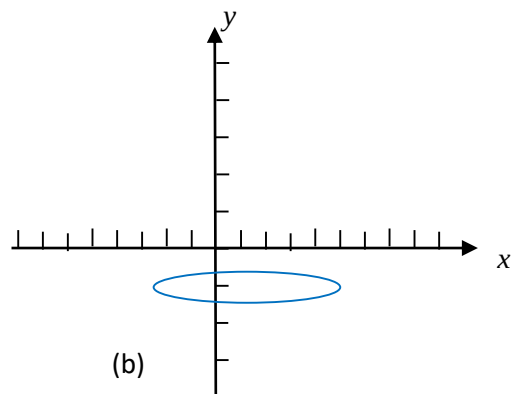
$$= \pm 2.2$$

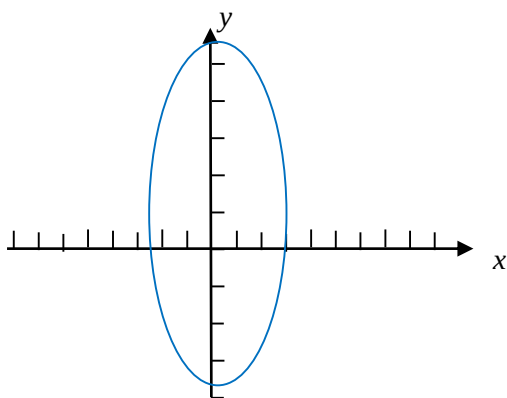
$$(0, \pm 2.2)$$

$$\text{Vertices : } \sqrt{a^2} = \sqrt{9} = \pm 3$$

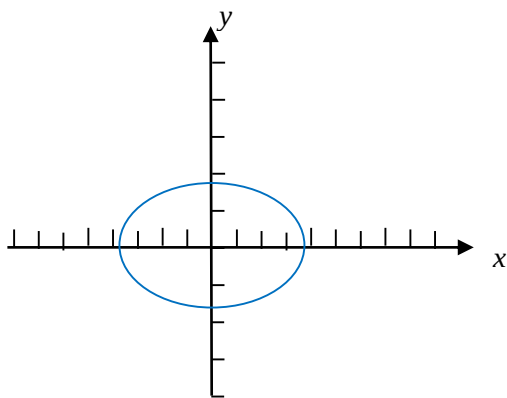
$$\sqrt{b^2} = \sqrt{4} = \pm 2$$

$$(\pm 2, 0) \quad (0, \pm 3)$$





(c)



(d)

7. $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (d)

8. $\frac{y^2}{36} + \frac{x^2}{9} = 1$ (c)

9. $\frac{(y-2)^2}{16} + \frac{(x+3)^2}{4} = 1$ (a)

10. $\frac{(x-1)^2}{11} + (y+2)^2 = 1$ (b)

In Exercises 1-6 find the vertices of the hyperbola.

1. $\frac{x^2}{16} - \frac{y^2}{7} = 1$

2. $\frac{y^2}{25} - \frac{x^2}{21} = 1$

$c = \sqrt{16 + 7} = \sqrt{23} = \pm 4.8$

$c = \sqrt{25 + 21} = \sqrt{46} = \pm 6.8$

Center : $(h, K) = (0,0)$

Center : $(0,0)$

Foci : $(h \pm c, K)$

Foci : $(h, K \pm c)$

$(0 \pm 4.8, 0)$

$(0, 0 \pm 6.8)$

$(\pm 4.8, 0)$

$(0, \pm 6.8)$

Vertices : $(h \pm a, K)$

Vertices : $(h, K \pm b)$

$(0 \pm 4, 0)$

$(0, 0 \pm 5)$

$(\pm 4, 0)$

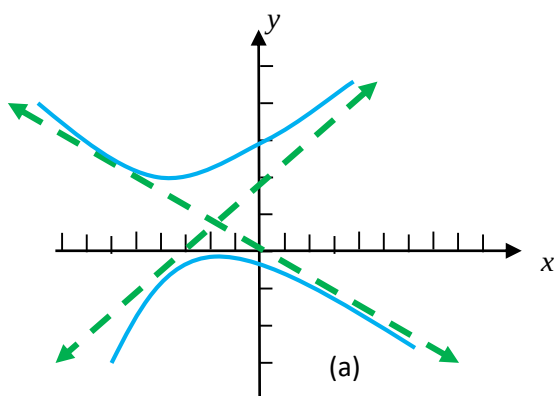
$(0, \pm 5)$

3. $\frac{y^2}{36} - \frac{x^2}{13} = 1$

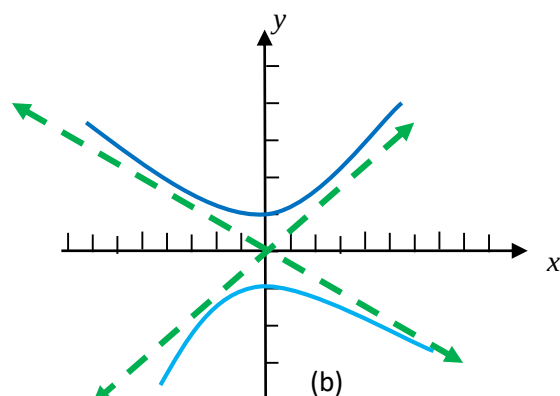
4. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

5. $3x^2 - 4y^2 = 12$

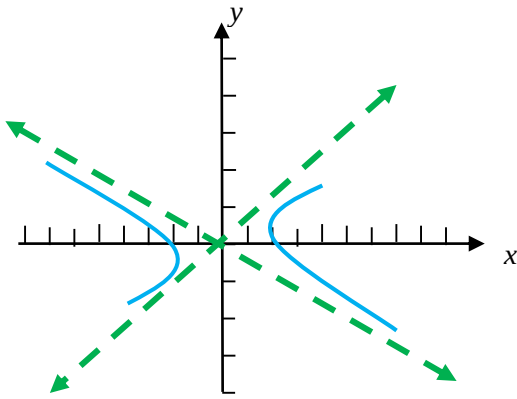
6. $9x^2 - 4y^2 = 36$



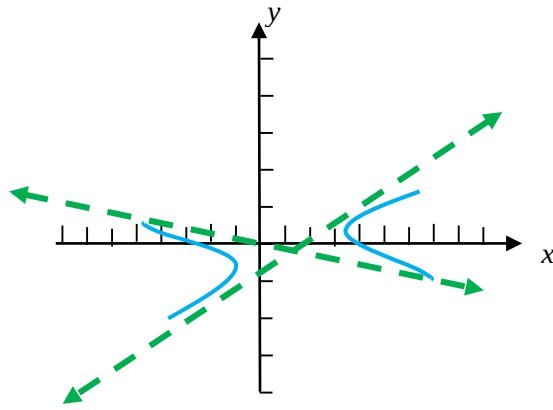
(a)



(b)



(c)



(d)

$$7. \frac{x^2}{25} - \frac{y^2}{16} = 1 \quad (c)$$

$$8. \frac{y^2}{4} - \frac{x^2}{9} = 1 \quad (b)$$

$$9. \frac{(y-2)^2}{4} - \frac{(x+3)^2}{16} = 1 \quad (a)$$

$$10. \frac{(x-2)^2}{9} - (y+1)^2 = 1 \quad (d)$$

In exercises 11-16, sketch the graph of the hyperbola by hand.

$$11. \frac{x^2}{49} - \frac{y^2}{25} = 1 \quad \text{L\&R}$$

$$12. \frac{y^2}{64} - \frac{x^2}{25} = 1 \quad \text{U\&D}$$

$$13. \frac{y^2}{25} - \frac{x^2}{16} = 1 \quad \text{U\&D}$$

$$14. \frac{x^2}{169} - \frac{y^2}{144} = 1 \quad \text{L\&R}$$

How can you determine which direction the graph will go ?

The graph direction is based on which leading variable is first.

$$13. \frac{y^2}{25} - \frac{x^2}{16} = 1$$

$$14. \frac{x^2}{169} - \frac{y^2}{144} = 1$$

$$15. \frac{(x+3)^2}{16} - \frac{(y-1)^2}{4} = 1$$

$$16. \frac{(x-1)^2}{2} - \frac{(y+3)^2}{4} = 1$$

In the exercises 39-42, find the center, vertices and the foci of the hyperbola.

$$39. \frac{(x+1)^2}{144} - \frac{(y-2)^2}{25} = 1$$

$$40. \frac{(x+4)^2}{12} - \frac{(y+6)^2}{13} = 1$$

$$41. \frac{(y+3)^2}{64} - \frac{(x-2)^2}{81} = 1$$

$$42. \frac{(y-1)^2}{25} - \frac{(x+5)^2}{11} = 1$$

Hyperbola

Standard Equation of a Hyperbola

The standard form of the equation of a hyperbola with center at (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Transverse axis is horizontal

or

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Transverse axis is vertical

The vertices are a units from the center, and the foci are c units from the center, where,

$$c^2 = a^2 + b^2.$$

Asymptotes of a Hyperbola

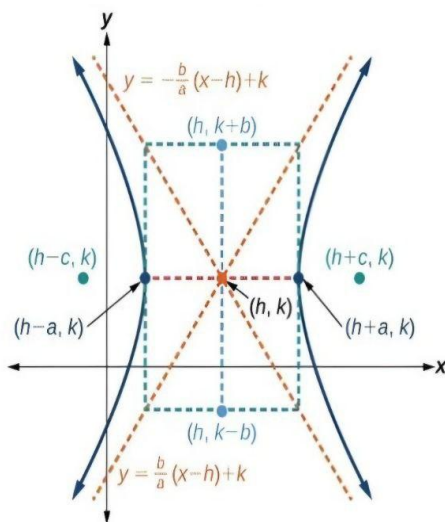
For a horizontal transverse axis, the equations of the asymptotes are

$$y = k + \frac{b}{a}(x-h) \quad \text{and} \quad y = k - \frac{b}{a}(x-h)$$

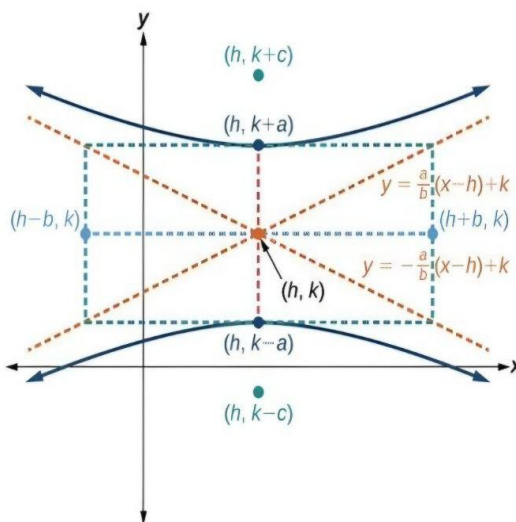
For a vertical transverse axis, the equations of the asymptotes are

$$y = k + \frac{a}{b}(x-h) \quad \text{and} \quad y = k - \frac{a}{b}(x-h)$$

$$(a) \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad (b) \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



(a)



(b)

find the vertices and foci of the ellipse.

$$1. \frac{x^2}{16} + \frac{y^2}{7} = 1$$

$$\text{Center: } \sqrt{16+7} = \sqrt{23} = \pm 4.8$$

$$\begin{aligned} \text{Foci: } (h \pm c, k) \\ (0 \pm 4.8, 0) \\ (\pm 4.8, 0) \end{aligned}$$

$$\begin{aligned} \text{Vertices: } (h \pm c, k) \\ (0 \pm 4, 0) \\ (\pm 4, 0) \end{aligned}$$

$$2. \frac{x^2}{25} + \frac{y^2}{21} = 1$$

$$\text{Center: } (0, 0)$$

$$\begin{aligned} \text{Foci: } C &= (h, k \pm c) \\ &= (0, 0 \pm 6.8) \\ &= (0, \pm 6.8) \end{aligned}$$

$$\begin{aligned} \text{Vertices: } (h, k \pm b) \\ (0, 0 \pm 5) \\ (0, \pm 5) \end{aligned}$$

Q1) The equations of the asymptotes are: $y = \pm \frac{\sqrt{7}}{4}x$.

Q2) The equations of the asymptotes are: $y = \pm \frac{5}{\sqrt{21}}x$.

$$3. \frac{y^2}{36} - \frac{x^2}{3} = 1$$

$$4. \frac{x^2}{9} - \frac{y^2}{16} = 1$$

Process of Finding Partial Fraction:

A proper fraction $\frac{N(x)}{D(x)}$ can be resolved into partial fractions as:

(I) If in the denominator $D(x)$ a linear factor $(ax + b)$ occurs and is non-repeating, its partial fraction will be of the form $\frac{A}{ax+b}$ where A is a constant whose value is to be determined.

(II) If in the denominator $D(x)$ a linear factor $(ax + b)$ occurs n times, i.e., $(ax + b)^n$, then there will be n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}$$

, where $A_1, A_2, A_3, \dots, A_n$ are constants, whose values are to be determined

(III) If in the denominator $D(x)$ a quadratic factor $ax^2 + bx + c$ occurs and is non-repeating, its partial fraction will be of the form

$$\frac{Ax+B}{ax^2+bx+c}, \text{ where } A \text{ and } B \text{ are constants, whose values are to}$$

be determined.

Example 1:

Resolve $\frac{7x-25}{(x-3)(x-4)}$ into partial fractions.

Solution:

$$\frac{7x-25}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4} \dots \dots \dots (1)$$

Multiplying both sides by L.C.M. i.e., $(x-3)(x-4)$, we get

$$7x-25 = A(x-4) + B(x-3) \dots \dots \dots (2)$$

Put $x-4=0, \Rightarrow x=4$ in equation (2)

$$7(4)-25 = A(4-4) + B(4-3)$$

$$28-25 = 0 + B(1)$$

$$B = 3$$

Put $x-3=0, \Rightarrow x=3$ in equation (2)

$$7(3) - 25 = A(3 - 4) + B(3 - 3)$$

$$21 - 25 = A(-1) + 0$$

$$A = 5$$

Hence the required partial fractions are

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x - 3} + \frac{3}{x - 4}$$

Resolve into partial fraction:

$$Q.1 \quad \frac{2x + 3}{(x - 2)(x + 5)}$$

$$Q.1 \quad \frac{1}{x - 2} + \frac{1}{x + 5}$$

$$Q.2 \quad \frac{2x + 5}{x^2 + 5x + 6}$$

$$Q.2 \quad \frac{1}{x + 2} + \frac{1}{x + 3}$$

Type II:

When the factors of the denominator are all linear but some are repeated.

Example 1:

Resolve into partial fractions: $\frac{x^2-3x+1}{(x-1)^2(x-2)}$

Solution:

$$\frac{x^2 - 3x + 1}{(x - 1)^2(x - 2)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x - 2}$$

Multiplying both sides by L.C.M. i.e., $(x - 1)^2(x - 2)$, we get

$$x^2 - 3x + 1 = A(x - 1)(x - 2) + B(x - 2) + C(x - 1)^2 \text{ (I)}$$

Putting $x - 1 = 0 \Rightarrow x = 1$ in (I), then

$$(1)^2 - 3(1) + 1 = B(1 - 2)$$

$$1 - 3 + 1 = -B$$

$$-1 = -B$$

$$\Rightarrow B = 1$$

Putting $x - 2 = 0 \Rightarrow x = 2$ in (I), then

$$(2)^2 - 3(2) + 1 = C(2 - 1)^2$$

$$4 - 6 + 1 = C(1)^2$$

$$\Rightarrow -1 = C$$

$$\text{Now } x^2 - 3x + 1 = A(x^2 - 3x + 2) + B(x - 2) + C(x^2 - 2x + 1)$$

Comparing the co-efficient of like powers of x on both sides, we get

$$\begin{aligned} A + C &= 1 \\ A &= 1 - C \\ &= 1 - (-1) \\ &= 1 + 1 = 2 \end{aligned}$$

$$\Rightarrow A = 2$$

Hence the required partial fractions are

$$\frac{x^2 - 3x + 1}{(x - 1)^2(x - 2)} = \frac{2}{x - 1} + \frac{1}{(x - 1)^2} + \frac{1}{x - 2}$$

Example 3:

Resolve into partial fractions $\frac{4+7x}{(2+3x)(1+x)^2}$

Solution:

$$\frac{4 + 7x}{(2 + 3x)(1 + x)^2} = \frac{A}{2 + 3x} + \frac{B}{1 + x} + \frac{C}{(1 + x)^2}$$

Multiplying both sides by L.C.M. i.e., $(2 + 3x)(1 + x)^2$

We get $4 + 7x = A(1 + x)^2 + B(2 + 3x)(1 + x) + C(2 + 3x) \dots (I)$

$$\text{Put } 2 + 3x = 0 \Rightarrow x = -\frac{2}{3} \text{ in (I)}$$

$$\text{Then } 4 + 7\left(-\frac{2}{3}\right) = A\left(1 - \frac{2}{3}\right)^2$$

$$4 - \frac{14}{3} = A\left(-\frac{1}{3}\right)^2$$

$$-\frac{3}{2} = \frac{1}{9}A$$

$$\Rightarrow A = -\frac{2}{3} \times \frac{9}{1} = -6$$

Put $1 + x = 0 \Rightarrow x = -1$ in eq. (I) we get

$$4 + 7(-1) = C(2 - 3)$$

$$4 - 7 = C(-1)$$

$$-3 = -C$$

$$\Rightarrow C = 3$$

$$4 + 7x = A(x^2 + 2x + 1) + B(2 + 5x + 3x^2) + C(2 + 3x)$$

Comparing the co-efficient of x^2 on both sides

$$\begin{aligned}
 A + 3B &= 0 \\
 -6 + 3B &= 0 \\
 3B &= 6
 \end{aligned}$$

$$\Rightarrow B = 2$$

Hence the required partial fraction will be

$$\frac{-6}{2+3x} + \frac{2}{1+x} + \frac{3}{(1+x)^2}$$

$$Q.1 \quad \frac{x+4}{(x-2)^2(x+1)}$$

$$Q.2 \quad \frac{1}{(x+1)(x^2-1)}$$

Type III:

When the denominator contains ir-reducible quadratic factors which are non-repeated.

Example 1:

Resolve into partial fractions $\frac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by L.C.M. i.e., $(x+3)(x^2+1)$, we get

$$9x-7 = A(x^2+1) + (Bx+C)(x+3) \quad (I)$$

Put $x+3=0 \Rightarrow x=-3$ in Eq. (I), we have

$$\begin{aligned} 9(-3)-7 &= A((-3)^2+1) + (B(-3)+C)(-3+3) \\ -27-7 &= 10A+0 \end{aligned}$$

$$A = -\frac{34}{10} \Rightarrow \boxed{A = -\frac{17}{5}}$$

$$9x-7 = A(x^2+1) + B(x^2+3x) + C(x+3)$$

Comparing the co-efficient of like powers of x on both sides

$$\begin{aligned} A+B &= 0 \\ 3B+C &= 9 \end{aligned}$$

Putting value of A in Eq. (i)

$$-\frac{17}{5} + B = 0 \Rightarrow \boxed{B = \frac{17}{5}}$$

From Eq. (iii)

$$\begin{aligned} C &= 9 - 3B = 9 - 3\left(\frac{17}{5}\right) \\ &= 9 - \frac{51}{5} \Rightarrow \boxed{C = -\frac{6}{5}} \end{aligned}$$

Hence the required partial fraction are

$$-\frac{17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

1.1 Limits at a point :

$\lim_{x \rightarrow a} f(x)$ is the value of $f(x)$ when x approaches to a .

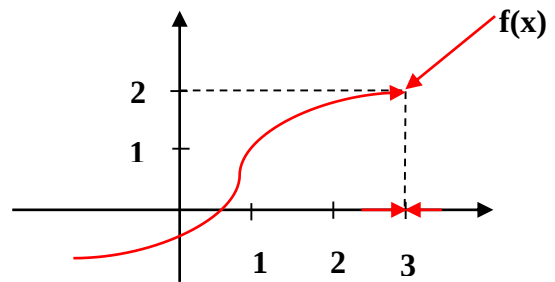
$\lim_{x \rightarrow a} c = c$ (Example $\lim_{x \rightarrow 3} 5 = 5$)

$\lim_{x \rightarrow a} bx + c = b \cdot a + c$ (Example: $\lim_{x \rightarrow 3} 5x + 2 = 5 \cdot 3 + 2 = 15 + 2 = 17$)

Example: Find the limit

- a) $\lim_{x \rightarrow 1} 3x^2 + 2x - 5 = 3 \cdot (1)^2 + 2 \cdot 1 - 5 = 3 + 2 - 5 = 0$
- b) $\lim_{x \rightarrow 4} \frac{3x-2}{5-2x} = \frac{3 \cdot 4 - 2}{5 - 2 \cdot 4} = \frac{10}{-3}$
- c) $\lim_{x \rightarrow -2} (2x + 1)^3 = (2 \cdot (-2) + 1)^3 = (-3)^3 = -27$
- d) $\lim_{x \rightarrow 2} \frac{x-2}{x+3} = \frac{0}{5} = 0$
- e) $\lim_{x \rightarrow 3} f(x) = 2$

$\Rightarrow$ when x approaches to 3 $f(x)$ approaches to 2



2.1 Limits at left and limits at right:

$\lim_{x \rightarrow a^+} f(x)$ is the limit of $f(x)$ at right of a as x approaches to a from the right, but actually we substitute by a .

$\lim_{x \rightarrow a^-} f(x)$ is the limit of $f(x)$ at left to a as x approaches to a from the left but actually we substitute by a

Note: $a^- \neq -a$

Example:

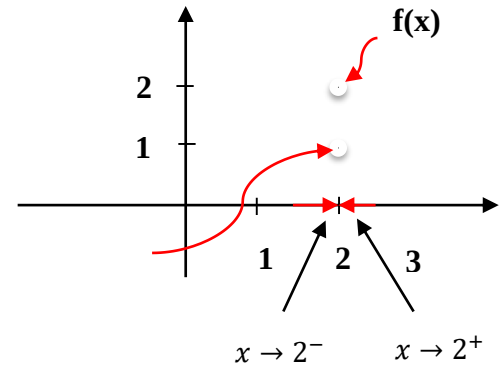
a) $\lim_{x \rightarrow 2^+} 5x + 3 = 5 \cdot 2 + 3 = 13$

b) $\lim_{x \rightarrow 2^+} 5x + 3 = 5 \cdot 2 + 3 = 13$

c) $\lim_{x \rightarrow 2^-} f(x) = 1$

$\lim_{x \rightarrow 2^+} f(x) = 2$

$\lim_{x \rightarrow a} f(x) = l$ (exist) if: $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = l$



Example: Find the limit at left and limit at right at a=3

a) $f(x) = 2x + 1$

$\lim_{x \rightarrow 3^+} 2x + 1 = 2 \cdot 3 + 1 = 7$

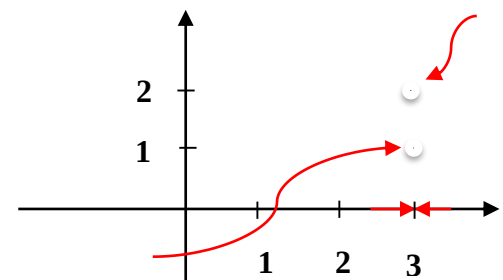
$\lim_{x \rightarrow 3^-} 2x + 1 = 2 \cdot 3 + 1 = 7$

$\Rightarrow \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = 7 \Rightarrow \lim_{x \rightarrow 3} f(x) = 7$ exist

b) $\lim_{x \rightarrow 3^-} f(x) = 1$

$\neq \{ \Rightarrow \lim_{x \rightarrow 3} f(x) = \text{not exist} \}$

$\lim_{x \rightarrow 3^+} f(x) = 2$



c) $f(x) = \frac{5}{x-3}$

$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \frac{5}{x-3} = \frac{5}{0^-} = -\infty$

$\{ \Rightarrow \lim_{x \rightarrow 3} f(x) = \text{not exist} \}$

$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \frac{5}{x-3} = \frac{5}{0^+} = +\infty$

Undefined form:

1. By factorization:

Example: Find the limits

a) $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} = \frac{2^2 + 2 - 6}{2^2 - 4} = \frac{0}{0}$ U.F

$\Rightarrow \lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{(x-2)(x+3)}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{x+3}{x+2} = \frac{5}{4}$

b) $\lim_{x \rightarrow -3} \frac{x^2 + 6x + 9}{2x^2 + 6x} = \frac{(-3)^2 + 6 \cdot (-3) + 9}{2 \cdot (-3)^2 + 6 \cdot (-3)} = \frac{0}{0}$ U.F

$$\lim_{x \rightarrow -3} \frac{x^2 + 6x + 9}{2x^2 + 6x} = \lim_{x \rightarrow -3} \frac{(x+3)^2}{2x(x+3)} = \lim_{x \rightarrow -3} \frac{x+3}{2x} = \frac{0}{-6} = 0$$

2. By multiplying by the conjugate:

Example:

$$\text{a) } \lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2} = \frac{4-4}{\sqrt{4}-2} = \frac{0}{0} \quad U.F$$

$$\Rightarrow \lim_{x \rightarrow 4} \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x \rightarrow 4} \frac{(x-4)(\sqrt{x}+2)}{\sqrt{x}^2 - 2^2} = \lim_{x \rightarrow 4} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x}-4}$$

$$= \lim_{x \rightarrow 4} \sqrt{x} + 2 = \sqrt{4} + 2 = 4$$

$$\text{b) } \lim_{x \rightarrow 3} \frac{\sqrt{x-2}-1}{x-3} = \frac{\sqrt{1}-1}{3-3} = \frac{0}{0} \quad U.F$$

$$\Rightarrow \lim_{x \rightarrow 3} \frac{(\sqrt{x-2}-1)(\sqrt{x-2}+1)}{(x-3)(\sqrt{x-2}+1)} = \lim_{x \rightarrow 3} \frac{\sqrt{x-2}^2 - 1^2}{(x-3)(\sqrt{x-2}+1)} = \lim_{x \rightarrow 3} \frac{x-2-1}{(x-3)(\sqrt{x-2}+1)} = \lim_{x \rightarrow 3} \frac{\cancel{x-3}}{(\cancel{x-3})(\sqrt{x-2}+1)}$$

$$\lim_{x \rightarrow 3} \frac{1}{(\sqrt{x-2}+1)} = \frac{1}{2}$$

3. Unifying the dominators:

Example:

$$\text{a) } \lim_{x \rightarrow 4} \frac{\frac{1}{4x} - \frac{1}{4}}{4-x} = \frac{\frac{1}{4 \cdot 4} - \frac{1}{4}}{4-4} = \frac{0}{0} \quad U.F$$

$$\Rightarrow \lim_{x \rightarrow 4} \frac{\frac{x}{4x} - \frac{4}{4x}}{4-x} = \lim_{x \rightarrow 4} \frac{\frac{x-4}{4x}}{4-x} = \lim_{x \rightarrow 4} \frac{x-4}{4x} \cdot \frac{1}{4-x}$$

$$= \lim_{x \rightarrow 4} \frac{x-4}{4x \cdot (4-x)} = \lim_{x \rightarrow 4} \frac{\cancel{x-4}}{4x \cdot (\cancel{4-x})} = \lim_{x \rightarrow 4} \frac{-1}{4x} = \frac{-1}{16}$$

Rules of Operations on Limits:

Let $f(x)$ and $g(x)$ be two defined functions on an interval containing a and if

$\lim_{x \rightarrow a} f(x) = h$ and $\lim_{x \rightarrow a} g(x) = k$, then:

- $\lim_{x \rightarrow a} [f(x) \pm g(x)] = h \pm k$
- $\lim_{x \rightarrow a} [f(x)g(x)] = h \cdot k$
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{h}{k}$, (if $k \neq 0$)

❖ Limits of Trigonometric Function:

Definition: $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

- When value of x approaches zero value of $\sin \theta$ approaches zero too.
($\sin \theta = \theta$)

Ex.17 Determine $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{2\theta}$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{2\theta} = \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\theta} \cdot \frac{1}{2} = \lim_{\theta \rightarrow 0} \frac{2\sin 2\theta}{2\theta} = 2 \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{2\theta} = (1)2=2$$

Ex.18 Determine $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{(1 + \cos x)} \\ &= 1 \left(\frac{0}{1 + 1} \right) = 0 \end{aligned}$$

Ex.19 Find the following limits:

- $\lim_{x \rightarrow 0} \frac{\tan x}{x}$
- $\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x}$

Limit of a Function at Infinity:

Rule:

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

There are three cases for the limit of a function when $x \rightarrow \infty$,

- | | | |
|--|-------------------------------------|--|
| 1. $f(x)$ may have a finite limit value. | 2. $f(x)$ may approach to infinity. | 3. $f(x)$ may oscillate or infinitely and limit value does not exist |
|--|-------------------------------------|--|

Limits of Rational Functions at Infinity:

- Divide the numerator and denominator by the highest power of x in the denominator.

Ex.20 Find limit of $f(x) = \frac{2x+4}{2x-3}$ when x goes to infinity

$$\lim_{x \rightarrow \infty} \frac{2x+4}{2x-3} = \lim_{x \rightarrow \infty} \frac{\frac{2x+4}{x}}{\frac{2x-3}{x}} = \lim_{x \rightarrow \infty} \frac{2 + \frac{4}{x}}{2 - \frac{3}{x}} = \frac{1+0}{1+0} = \frac{2}{2} = 1$$

Note: when the degree of numerator and denominator are same so the limit equal division of their coefficient.

$$\text{Ex.21 } \lim_{x \rightarrow \infty} \frac{-x^3+4x}{2x^2-5x-3} = \lim_{x \rightarrow \infty} \frac{\frac{-x^3+4x}{x^3}}{\frac{2x^2-5x-3}{x^3}} = \lim_{x \rightarrow \infty} \frac{\frac{1+\frac{4}{x^2}}{x}}{\frac{2}{x} - \frac{5}{x^2} - \frac{3}{x^3}} = \frac{1+0}{0+0+0} = \frac{1}{0} = \infty$$

Note: when the degree of numerator higher than the denominator the limit equal ∞ * (sign leading coefficient numerator / sign leading coefficient of denominator)

$$\text{Ex.21 } \lim_{x \rightarrow \infty} \frac{x^2-3x+1}{2x^3-6} = \lim_{x \rightarrow \infty} \frac{\frac{x^2-3x+1}{x^3}}{\frac{2x^3-6}{x^3}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{3}{x^2} + \frac{1}{x^3}}{2 - \frac{6}{x^3}} = \frac{0-0+0}{2-0} = \frac{0}{2} = 0$$

Note: when the degree of denominator higher than the degree numerator the limit equal zero.

Continuity:

Definition:

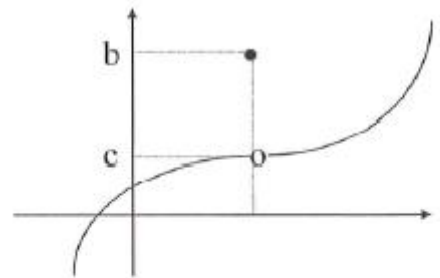
A function $f(x)$ is **continuous** at $x=a$, if and only if

- (a) $f(x)$ is well-defined at $x=a$, $f(a)$ exists, and $f(a)$ is a finite value,
- (b) $\lim_{x \rightarrow a} f(x)$ exists and $\lim_{x \rightarrow a} f(x) = f(a)$.

Otherwise, the function is **discontinuous** at $x=a$.

$\lim_{x \rightarrow a} f(x)$ exists but not equal to $f(a)$.

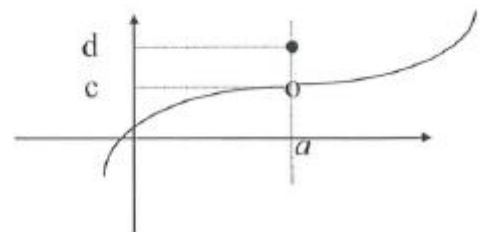
$$\lim_{x \rightarrow a} f(x) = c$$



$$\lim_{x \rightarrow a^-} f(x) = c$$

$$\lim_{x \rightarrow a^+} f(x) = d$$

$$\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$$



Piecewise Function

Ex.24 Let $f(x) = \begin{cases} 3 & \text{if } x \neq 5 \\ 1 & \text{if } x = 5 \end{cases}$, then

$$\lim_{x \rightarrow 5} f(x) = 3 \text{ but } f(5) = 1,$$

So $\lim_{x \rightarrow 5} f(x) \neq f(5)$. It is **discontinuous**

Ex.25 Consider the function $f(x) = \begin{cases} 2x + 1, & x \neq 2 \\ 4, & x = 2 \end{cases}$

$f(x)$ is discontinuous at $x = 2$ since $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (2x + 1) = 5 \neq f(2)$.

Ex.26 Show that the function $f(x) = \begin{cases} 2x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$ is continuous at $x = 0$.

Ex.27 Show that $f(x) = \begin{cases} 2x + 1, & x \neq 2 \\ 4, & x = 2 \end{cases}$ is discontinuous at $x = 2$.

Solution: Since $f(2) = 4$ and $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (2x + 1) = 5 \neq f(2)$

So $f(x)$ is discontinuous at $x = 2$.

Ex.28 Show that $f(x) = \begin{cases} 2x + 1, & x \leq 2 \\ x^3 - 5, & x > 2 \end{cases}$ is discontinuous at $x = 2$

Solution: Since $f(2) = 4$ and $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (2x + 1) = 5 \neq f(2)$,

So $f(x)$ is discontinuous at $x = 2$

Ex.29 Find the value of k that would make the function continuous in each case.

$$f(x) = \begin{cases} \frac{x^2-9}{x+3} & x \neq -3 \\ k & x = -3 \end{cases}$$