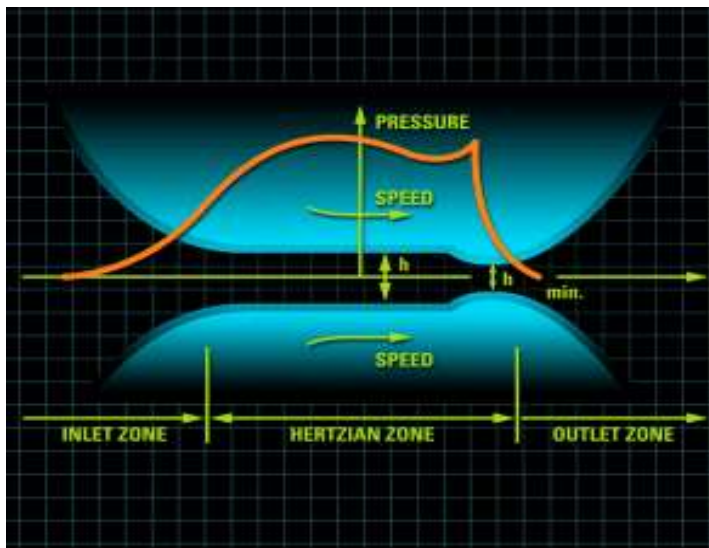
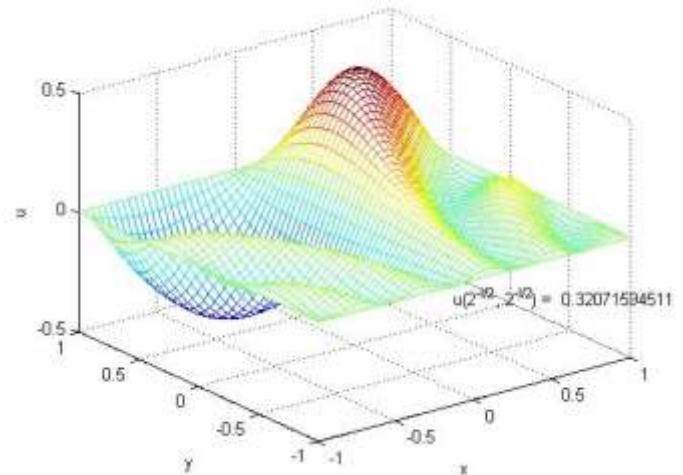


Numerical Analysis-1



Solution of 2D-Poisson Problem: $u_{xx} + u_{yy} = 10 \sin(8x(y-1))$, $u = 0$ on Boundary



Dr Hashim Khan

Lecture Notes: (MATH-316)

Errors and their computations, Numerical solution of systems of linear algebraic equations, Numerical solution of nonlinear equations, Interpolation, Numerical differentiation and integration, Numerical solution of ordinary differential equations

Numerical Analysis-1

Course Teaching Plan:

MATH-316

Week-Wise Lecture Plan:

[For All Campuses]

Chapter: Title	Topic	Assessment task	Due Week
Chapter-1: Numerical Solution of Systems of Linear Algebraic Equations	1.1 Errors and their computations		1 st week
	1.2 Test for convergence for iterative		
	1.3 Gauss elimination method		
	1.4 Gauss Jordan method	HW-1	2 nd week
	1.5 LU Factorization method		
Chapter-2: Numerical Solution of Nonlinear Equations	1.6 Gauss-Jacobi method		3 rd week
	1.7 Gauss Seidel method		
	2.1 Nonlinear Equations		4 th week
	2.2 Bisection method		
	2.3 Newton-Raphson method		
Chapter-3: Interpolation	2.4 Secant method		5 th week
	2.5 Nonlinear system of equations		
	3.1 Finite difference operators	HW-2	6 th week
	3.2 Important relations between operators		
	3.3 Newton's forward & backward interpolation formula		
Chapter-4: Numerical Differentiation and Integration	3.4 Lagrange's interpolation polynomial formula	First Midterm Exam	7 th week
	3.5 Newton's divided difference interpolation polynomial formula		
	3.6 Inverse interpolation		
	4.1 Numerical differentiation and its application		8 th week
	4.2 Newton-Cote's integration formula		
Chapter-5: Numerical Solution of Ordinary Differential Equations	4.3 Trapezoidal rule		9 th week
	4.4 Simpson's- 1/3 rule		
	4.5 Simpson's- 3/8 rule		
	5.1 Euler's & Modified Euler's method	Second Midterm Exam	10 th week
	5.2 Taylor's series method		
	5.3 Heun's (Runge-Kutta 2 nd order) method		
	5.4 Runge-Kutta 3 rd order method		11 th week
	5.5 Runge-Kutta 4 th order method		
		Final Exam	12 st week

Grades Breakdown: (Liable to change in case of online exam)

Assessment task	Week Due	Proportion of Total Assessment
1- Homework + Quizzes	2 nd & 6 th	10%
2- First Mid-Term Exam	7 th or 8 th	20%
3- Second Mid-Term Exam	10 th or 11 th	20%
4- Final Exam	As scheduled	50%

Prescribed Textbook:

Book Name	Authors	Edition
Introduction to Numerical Methods	Autar Kaw & Kalu	http://numericalmethods.eng.usf.edu
Numerical Analysis	J. Douglas Faires, Richard Burden & Annette Burden	9 nd Edition An International Thomson Company

Scientific References:

Book Name	Authors	Edition
Introductory Methods of Numerical Analysis	S. S. Sastry	4 th Edition, Prentice Hall of India
Numerical methods for scientific & engineering computation	M K Jain, S R K Iyenger, R K Jain	John Wiley & Sons Canada
Numerical methods for scientists & engineers	R W Hamming	2 nd Edition ISBN-10:0486652416
Numerical mathematics and computing	W Cheney and D Kincaid	6 th Edition, Bob Pirtle Publisher

Important Note:

- First Mid-Term Exam will be held on 7th/8th week. The date of exam will be announced soon.
- Second Mid-Term Exam will be held on 10th/11th week. The date of exam will be announced soon.

Learning Outcomes:

Upon successful completion of the course, the student is expected to be familiar with the following:

- Understand and grasp the deep knowledge of Numerical analysis
- Application of Numerical methods for solving a variety of problems
- Some Software in Applied Numerical Method
- Develop logical thinking and ability to apply them in solving mathematical problems involved in technical specialty courses.

Course Coordinator

Numerical Analysis-1

(For Science College Students)

MATH-316

Chapter-1: Numerical Solution of Systems of Linear Algebraic Equations

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Contents:

Chapter-1: Numerical Solution of Systems of Linear Algebraic Equations

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1.1 Errors and Their Computations:

Let us first understand the following fundamentals of numbers-

(a) Numbers: There are two kinds of numbers:

(i) Exact Number: The exact numbers are written in the following manner-

$1, 2, 3, \dots, 1/2, 3/2, \dots, \sqrt{2}, \sqrt{3}, \dots, \pi$ etc.

(ii) Approximate Number: The approximate numbers are those that represent the numbers to a certain degree of accuracy.

Example: An approximate value of π is 3.1416, or if we desire a better approximation, it is 3.14159265. But we cannot write the exact value of π .

(b) Significant Digits: The digits that are used to express a number are called significant digits or significant figures. Significant digits are written as the beginning with the leftmost nonzero digit and ending with the rightmost correct digit, including final zeros that are exact.

Number	Number of Significant Digits
3.1416	5
0.66667	5
4.0687	5
0.00023	2
0.00145	3
0.000145	3
0.0000145	3
25600	Uncertain
2.56×10^4	3
2.560×10^4	4
2.5600×10^4	5

(c) Round-off: To round-off a number to n significant digits, discard all digits to the right of the n th digits, and if this discarded number is:

- (i)** Less than half a unit in the n th place, leave the n th digit unaltered
- (ii)** Greater than half a unit in the n th place, increase the n th digit by unity
- (iii)** Exactly half a unit in the n th place, increase the n th digits by unity if it is odd, otherwise, leave it unchanged.

The number, thus rounded-off is said to be correct to n significant figures.

(d) Accuracy and Precision: Accuracy to n decimal places means that you can trust n digits to the right of the decimal place.

Accurate to n significant digits means that you can trust a total of n digits as being meaningful beginning with the leftmost nonzero digit.

(e) Source of Errors:

- (i) Round-off Error:** The round-off error is caused by rounding-off a number as discussed above. The round-off error can be reduced by carrying out the computations to more significant figures at each step of the computation.
- (ii) Truncation Error:** The truncation error can be caused by using approximate formula in computations such as the one that arises when a truncated infinite series is used. The study of truncation error is naturally associated with the problem of convergence.

(f) Classification of Errors:

- (i) True Error:** True Error (E_T) is the numerical difference between the **Exact (True) Value (p)** of a quantity and its **Approximate Value (p_n)**.

$$\text{True Error: } E_T = |p - p_n| \quad \dots (1)$$

- (ii) Relative True Error:** The Relative Error (E_R) is defined by-

$$\text{Relative True Error: } E_R = \frac{|p - p_n|}{|p|} = \frac{E_T}{|p|} \quad \dots (2)$$

- (iii) Percentage True Error:** The Percentage Error (E_P) is defined by-

$$\text{Percentage True Error } (E_P) = \frac{|p - p_n|}{|p|} \cdot 100 = E_R \cdot 100 \quad \dots (3)$$

- (iv) Approximate Error:** Approximate Error (E_A) is the numerical difference between the **New Approximate Value (p_{new})** of a quantity and its **Old Approximate Value (p_{old})**.

$$\text{Approximate Error } (E_A) = |p_{new} - p_{old}| \quad \dots (4)$$

- (v) Relative Approximate Error:** The Relative Approximate Error (E_R) is defined by-

$$\text{Relative Approximate Error } (E_R) = \frac{|p_{new} - p_{old}|}{|p_{new}|} = \frac{E_A}{|p_{new}|} \quad \dots (5)$$

- (vi) Percentage Approximate Error:** The Percentage Approximate Error (E_P) is defined by-

$$\text{Percentage Approximate Error } (E_P) = \frac{|p_{new} - p_{old}|}{|p_{new}|} \cdot 100 = E_R \cdot 100 \quad \dots (6)$$

Note: All types of Errors defined above are calculated in the absolute form.

1.2 Systems of Linear Algebraic Equations:

Linear systems of equations are associated with many problems in engineering and science, as well as with applications of mathematics to the social sciences and the quantitative study of business and economic problems.

In this chapter we consider *direct methods* for solving a linear system of n equations in n variables. Such a system has the form

$$\begin{aligned} E_1 : \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ E_2 : \quad & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ & \vdots \\ E_n : \quad & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n. \end{aligned}$$

In this system we are given the constants a_{ij} , for each $i, j = 1, 2, \dots, n$, and b_i , for each $i = 1, 2, \dots, n$, and we need to determine the unknowns $x_1, \dots, x_n$.

[A] Direct Methods (Exact Methods):

- + Gauss-Elimination Method
- + Gauss-Jordan Method
- + LU-Factorization Method

Note: These numerical methods provide with **Exact Solutions** for a system of linear algebraic equations.

[B] Iterative Methods:

- + Gauss-Jacobi Iterative Method
- + Gauss-Seidel Iterative Method

Note: These numerical methods provide with **Approximate Solutions** for a system of linear algebraic equations.

1.3 Test for Convergence for Iterative Methods:

Let us define for $k = 0, 1, 2, 3, 4, \dots, n$ iterations-

$$\|x^{(k+1)}\|_{\infty} = \text{Maximum of } |x^{(k+1)}|$$

$$\|x^{(k+1)} - x^{(k)}\|_{\infty} = \text{Maximum of } |x^{(k+1)} - x^{(k)}|$$

Working Procedure:**Step-1:** We have augmented matrix –

$$[A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} : b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} : b_2 \\ \vdots & \vdots & \ddots & \vdots : \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} : b_n \end{bmatrix} \quad \dots (4)$$

We divide the first row of augmented matrix ... (4) by a_{11} (**pivot element**) and then subtract this row multiplied $a_{21}, a_{31} \dots, a_{n1}$ from $2^{\text{nd}}, 3^{\text{rd}}, \dots, n^{\text{th}}$ rows.

In symbolic form, we can write as follows:

$$R_2 \rightarrow R_2 - a_{21}R_1, \quad R_3 \rightarrow R_3 - a_{31}R_1 \quad \text{and} \quad R_n \rightarrow R_n - a_{n1}R_1]$$

The augmented matrix reduces to the form-

$$[A : B] \cong \begin{bmatrix} 1 & a'_{12} & \cdots & a'_{1n} : b'_1 \\ 0 & a'_{22} & \cdots & a'_{2n} : b'_2 \\ \vdots & \vdots & \ddots & \vdots : \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} : b'_n \end{bmatrix} \quad \dots (5)$$

Note: x_1 has eliminated from the last $(n - 1)$ equations.

This process is repeated with the second row of equivalent augmented matrix ... (5) *i.e.* We now divide the 2^{nd} row of equivalent augmented matrix ... (5) by a'_{22} (new pivot element) and then subtract this row multiplied $a'_{32}, a'_{42}, \dots, a'_{n2}$ from $3^{\text{rd}}, 4^{\text{th}}, \dots, n^{\text{th}}$ rows.

In symbolic form, we can write as follows:

$$R_3 \rightarrow R_3 - a'_{32}R_2, \quad R_4 \rightarrow R_4 - a'_{42}R_2 \quad \text{and} \quad R_n \rightarrow R_n - a'_{n2}R_2$$

The same process is repeated again and again till the given system assumes the upper triangular form:

$$\left. \begin{aligned} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \cdots + c_{2n}x_n &= d_2 \\ &\vdots \\ c_{nn}x_n &= d_n \end{aligned} \right\} \quad \dots (6)$$

Step-2: Now the values of unknowns are determined by back substitution procedure.

Example 1.2.1: Use **Gauss-Elimination Method** to solve the following linear system of equations:

$$\left. \begin{aligned} 5x + 2y + z &= -12 \\ 2x - 3y + 10z &= 3 \\ -x + 4y + 2z &= 20 \end{aligned} \right\}$$

Solution: Re-writing given linear system of equations as follows:

$$\left. \begin{aligned} x - 4y - 2z &= -20 \\ 2x - 3y + 10z &= 3 \\ 5x + 2y + z &= -12 \end{aligned} \right\}$$

The linear system of equations can be expressed in matrix form-

$$AX = B \quad \dots (1)$$

$$\begin{bmatrix} 1 & -4 & -2 \\ 2 & -3 & 10 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -20 \\ 3 \\ -12 \end{bmatrix} \quad \dots (2)$$

Step-1: Writing augmented matrix as follows-

$$[A : B] = \begin{bmatrix} 1 & -4 & -2 & : & -20 \\ 2 & -3 & 10 & : & 3 \\ 5 & 2 & 1 & : & -12 \end{bmatrix} \quad \dots (3)$$

Perform the following operations:

$$\begin{aligned} R_2 &\rightarrow R_2 - 2R_1 & 2R_1 &= [2 \quad -8 \quad -4 \quad -40] \\ R_3 &\rightarrow R_3 - 5R_1 & 5R_1 &= [5 \quad -20 \quad -10 \quad -100] \end{aligned}$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & -4 & -2 & : & -20 \\ 0 & 5 & 14 & : & 43 \\ 0 & 22 & 11 & : & 88 \end{bmatrix}$$

Perform the following operation:

$$R_2 \rightarrow \frac{1}{5}R_2$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & -4 & -2 & : & -20 \\ 0 & 1 & \frac{14}{5} & : & \frac{43}{5} \\ 0 & 22 & 11 & : & 88 \end{bmatrix}$$

Perform the following operation

$$R_3 \rightarrow R_3 - 22R_2$$

$$22R_2 = \begin{bmatrix} 0 & 22 & \frac{308}{5} & \frac{946}{5} \end{bmatrix}$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & -4 & -2 & : & -20 \\ 0 & 1 & \frac{14}{5} & : & \frac{43}{5} \\ 0 & 0 & -\frac{253}{5} & : & -\frac{506}{5} \end{bmatrix}$$

Perform the following operation:

$$R_3 \rightarrow \left(\frac{-5}{253}\right) R_3$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & -4 & -2 & : & -20 \\ 0 & 1 & \frac{14}{5} & : & \frac{43}{5} \\ 0 & 0 & 1 & : & 2 \end{bmatrix}$$

Step-2: Applying back substitution method: Writing first in matrix form as follows:

$$AX = B$$

$$\begin{bmatrix} 1 & -4 & -2 \\ 0 & 1 & \frac{14}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -20 \\ \frac{43}{5} \\ 2 \end{bmatrix}$$

$$\Rightarrow \left. \begin{aligned} 1x - 4y - 2z &= -20 \\ 0x + 1y - \frac{14}{5}z &= \frac{43}{5} \\ 0x + 0y + 1z &= 2 \end{aligned} \right\}$$

$$\Rightarrow z = 2$$

$$y - \frac{14}{5}z = \frac{43}{5} \Rightarrow y = \frac{43}{5} - \frac{14}{5}(2) = 3$$

And

$$1x - 4y - 2z = -20$$

$$\Rightarrow x - 4(3) - 2(2) = -20$$

$$\Rightarrow x = -20 + 4(3) + 2(2) = -4$$

$$\Rightarrow x = -4, \quad y = 3, \quad z = 2$$

Example 1.2.2: Use Gauss Elimination Method to solve the following linear system of equations-

$$\left. \begin{aligned} 1x_1 + 2x_2 + 3x_3 + 4x_4 &= 10 \\ 7x_1 + 10x_2 + 5x_3 + 2x_4 &= 40 \\ 13x_1 + 6x_2 + 2x_3 - 3x_4 &= 34 \\ 11x_1 + 14x_2 + 8x_3 - 1x_4 &= 64 \end{aligned} \right\} \quad \dots (1)$$

Solution: The given linear system of equations can be expressed in matrix form-

$$AX = B \quad \dots (2)$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 7 & 10 & 5 & 2 \\ 13 & 6 & 2 & -3 \\ 11 & 14 & 8 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 40 \\ 34 \\ 64 \end{bmatrix} \quad \dots (3)$$

Step-1: Writing augmented matrix-

$$[A : B] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 7 & 10 & 5 & 2 & 40 \\ 13 & 6 & 2 & -3 & 34 \\ 11 & 14 & 8 & -1 & 64 \end{array} \right] \quad \dots (4)$$

Performing the following operations:

$$R_2 \rightarrow R_2 - 7R_1$$

$$R_3 \rightarrow R_3 - 13R_1$$

$$\text{And} \quad R_4 \rightarrow R_4 - 11R_1$$

We get the equivalent augmented matrix-

$$[A : B] \cong \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & -4 & -16 & -26 & -30 \\ 0 & -20 & -37 & -55 & -96 \\ 0 & -8 & -25 & -45 & -46 \end{array} \right]$$

Performing the following operations:

$$R_2 \rightarrow \frac{1}{-4}R_2$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 2 & 3 & 4 & \vdots & 10 \\ 0 & 1 & 4 & \frac{13}{2} & \vdots & \frac{15}{2} \\ 0 & -20 & -37 & -55 & \vdots & -96 \\ 0 & -8 & -25 & -45 & \vdots & -46 \end{bmatrix}$$

Performing the following operations:

$$R_3 \rightarrow R_3 + 20R_2$$

And $R_4 \rightarrow R_4 + 8R_2$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 2 & 3 & 4 & \vdots & 10 \\ 0 & 1 & 4 & \frac{13}{2} & \vdots & \frac{15}{2} \\ 0 & 0 & 43 & 75 & \vdots & 54 \\ 0 & 0 & 7 & 7 & \vdots & 14 \end{bmatrix}$$

Performing the following operation:

$$R_3 \rightarrow \frac{1}{43}R_3$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 2 & 3 & 4 & \vdots & 10 \\ 0 & 1 & 4 & \frac{13}{2} & \vdots & \frac{15}{2} \\ 0 & 0 & 1 & \frac{75}{43} & \vdots & \frac{54}{43} \\ 0 & 0 & 7 & 7 & \vdots & 14 \end{bmatrix}$$

Performing the following operations:

$$R_4 \rightarrow R_4 - 7R_3$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 4 & \frac{13}{2} & \frac{15}{2} \\ 0 & 0 & 1 & \frac{75}{43} & \frac{54}{43} \\ 0 & 0 & 0 & \frac{-224}{43} & \frac{224}{43} \end{bmatrix} \quad \dots (5)$$

Writing Equivalent linear system of equations:

$$\left. \begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 10 \\ x_2 + 4x_3 + \frac{13}{2}x_4 &= \frac{15}{2} \\ x_3 + \frac{75}{43}x_4 &= \frac{54}{43} \\ \frac{-224}{43}x_4 &= \frac{224}{43} \end{aligned} \right\} \quad \dots (6)$$

Step-2: Solving the above linear system of equations using back substitution method-

$$x_4 = \frac{\frac{224}{43}}{\frac{-224}{43}} \Rightarrow x_4 = -1$$

$$x_3 + \frac{75}{43}(-1) = \frac{54}{43}$$

$$\Rightarrow x_3 = \frac{54}{43} + \frac{75}{43} \Rightarrow x_3 = \frac{129}{43} \Rightarrow x_3 = 3$$

$$x_2 + 4(3) + \frac{13}{2}(-1) = \frac{15}{2}$$

$$\Rightarrow x_2 = \frac{15}{2} + \frac{13}{2} - 12 \Rightarrow x_2 = 2$$

$$x_1 + 2(2) + 3(3) + 4(-1) = 10$$

$$\Rightarrow x_1 = 10 - 4 - 9 + 4 \Rightarrow x_1 = 1$$

1.5 Pivoting, Partial Pivoting:

The Gauss Elimination method fails if one of the pivot elements (a_{ii}) becomes zero. In such situation, we rewrite the system of equations in a different order to avoid zero pivots. Thus, changing the order of equations is called pivoting.

In partial pivoting technique, if the pivot element a_{ii} happens to be zero then the i^{th} column element is searched for the numerically largest. Let the j^{th} row ($j > i$) contains this element then we interchange the i^{th} equation with the j^{th} equation and proceed for elimination.

This process is continued whenever pivots become zero during elimination.

1.6 Gauss-Jordan Method:

This method is a modified form of Gauss Elimination method. In Gauss-Jordan method, the coefficient matrix is reduced to a **diagonal matrix**. The partial pivoting is also used whenever the pivot element becomes zero.

Example 1.4.1: Use Gauss-Jordan Method to solve the following linear system of equations-

$$\left. \begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ x + y + 5z &= 7 \end{aligned} \right\} \quad \dots (1)$$

Solution: The given linear system of equations can be expressed in matrix form-

$$AX = B \quad \dots (2)$$

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix} \quad \dots (3)$$

Step-1: Writing augmented matrix-

$$[A : B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array} \right] \quad \dots (4)$$

Performing the following operations:

$$R_1 \rightarrow \frac{1}{10} R_1$$

We get the equivalent augmented matrix-

$$[A : B] \cong \left[\begin{array}{ccc|c} 1 & \frac{1}{10} & \frac{1}{10} & \frac{12}{10} \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array} \right]$$

Performing the following operations:

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$2R_1 = \left[2 \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{12}{5} \right]$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \vdots & \frac{12}{10} \\ 0 & \frac{49}{5} & \frac{4}{5} & \vdots & \frac{53}{5} \\ 0 & \frac{9}{10} & \frac{49}{10} & \vdots & \frac{58}{10} \end{bmatrix}$$

Performing the following operations:

$$R_2 \rightarrow \frac{5}{49} R_2$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \vdots & \frac{12}{10} \\ 0 & 1 & \frac{4}{49} & \vdots & \frac{53}{49} \\ 0 & \frac{9}{10} & \frac{49}{10} & \vdots & \frac{58}{10} \end{bmatrix}$$

Performing the following operations:

$$R_1 \rightarrow R_1 - \frac{1}{10} R_2$$

$$\frac{1}{10} R_2 = \begin{bmatrix} 0 & \frac{1}{10} & \frac{4}{490} & \frac{53}{490} \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{9}{10} R_2$$

$$\frac{9}{10} R_2 = \begin{bmatrix} 0 & \frac{9}{10} & \frac{36}{490} & \frac{477}{490} \end{bmatrix}$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 0 & \frac{9}{98} & \vdots & \frac{107}{98} \\ 0 & 1 & \frac{4}{49} & \vdots & \frac{53}{49} \\ 0 & 0 & \frac{473}{98} & \vdots & \frac{473}{98} \end{bmatrix}$$

Performing the following operations:

$$R_3 \rightarrow \frac{98}{473} R_3$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 0 & \frac{9}{98} & \vdots & \frac{107}{98} \\ 0 & 1 & \frac{4}{49} & \vdots & \frac{53}{49} \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix} \quad \dots (5)$$

Performing the following operations:

$$R_1 \rightarrow R_1 - \frac{9}{98} R_3$$

$$\frac{9}{98} R_3 = \begin{bmatrix} 0 & 0 & \frac{9}{98} & \frac{9}{98} \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \frac{4}{49} R_3$$

$$\frac{4}{49} R_3 = \begin{bmatrix} 0 & 0 & \frac{4}{49} & \frac{4}{49} \end{bmatrix}$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix} \quad \dots (6)$$

Writing Equivalent linear system of equations:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \dots (7)$$

$$\Rightarrow \quad x = 1, \quad y = 1, \quad z = 1$$

Example 1.4.2: Use **Gauss-Jordan Method** to solve the following linear system of equations:

$$\left. \begin{array}{l} x_1 + 4x_2 + 9x_3 = 16 \\ 3x_1 + 2x_2 + 3x_3 = 18 \\ 2x_1 + x_2 + x_3 = 10 \end{array} \right\} \quad \dots (1)$$

Solution: The linear system of equations can be expressed in matrix form-

$$AX = B \quad \dots (2)$$

$$\begin{bmatrix} 1 & 4 & 9 \\ 3 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 18 \\ 10 \end{bmatrix} \quad \dots (3)$$

Step-1: Writing **augmented matrix** as follows-

$$[A : B] = \begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 3 & 2 & 3 & : & 18 \\ 2 & 1 & 1 & : & 10 \end{bmatrix} \quad \dots (4)$$

Performing the following operations:

$$R_2 \rightarrow R_2 - 3R_1$$

$$3R_1 = [3 \quad 12 \quad 27 \quad 48]$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$2R_1 = [2 \quad 8 \quad 18 \quad 32]$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 0 & -10 & -24 & : & -30 \\ 0 & -7 & -17 & : & -22 \end{bmatrix}$$

Performing the following operations:

$$R_2 \rightarrow \frac{1}{-10} R_2$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 0 & 1 & \frac{12}{5} & : & 3 \\ 0 & -7 & -17 & : & -22 \end{bmatrix}$$

Performing the following operations:

$$R_1 \rightarrow R_1 - 4R_2$$

$$4R_2 = \begin{bmatrix} 0 & 4 & \frac{48}{5} & 12 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 7R_2$$

$$7R_2 = \begin{bmatrix} 0 & 7 & \frac{84}{5} & 21 \end{bmatrix}$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 0 & -\frac{3}{5} & \vdots & 4 \\ 0 & 1 & \frac{12}{5} & \vdots & 3 \\ 0 & 0 & -\frac{1}{5} & \vdots & -1 \end{bmatrix}$$

Performing the following operations:

$$R_3 \rightarrow -5R_3$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 0 & -\frac{3}{5} & \vdots & 4 \\ 0 & 1 & \frac{12}{5} & \vdots & 3 \\ 0 & 0 & 1 & \vdots & 5 \end{bmatrix}$$

Performing the following operations:

$$R_1 \rightarrow R_1 + \frac{3}{5}R_3$$

$$\frac{3}{5}R_3 = \begin{bmatrix} 0 & 0 & \frac{3}{5} & 3 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \frac{12}{5}R_3$$

$$\frac{12}{5}R_3 = \begin{bmatrix} 0 & 0 & \frac{12}{5} & 12 \end{bmatrix}$$

We get the equivalent augmented matrix-

$$[A : B] \cong \begin{bmatrix} 1 & 0 & 0 & \vdots & 7 \\ 0 & 1 & 0 & \vdots & -9 \\ 0 & 0 & 1 & \vdots & 5 \end{bmatrix}$$

Step-2: This can be expressed in matrix form as: $AX = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = 7, \quad x_2 = -9, \quad x_3 = 5$$

1.7 LU-Factorization Method: Before the start of LU-Factorization Method, let us understand the following terminology:

(a) Upper Triangular Matrix: A square matrix U is said to be **upper triangular matrix** if the element below of the **main diagonal** are zero.

e.g. The matrix $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ is called **upper triangular matrix**.

(b) Lower Triangular Matrix: A square matrix L is said to be **lower triangular matrix** if the element above of the **main diagonal** are zero.

e.g. The matrix $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ is called **lower triangular matrix**.

e.g. The matrix $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ is called **unit-lower triangular matrix**.

(c) LU-Factorization of a Matrix: We outline below the procedure for finding L and U for a given square matrix A of order 3.

We write as $A = L \cdot U$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \dots (1)$$

Multiplying the matrices on the right hand sides of equation... (1) and equating the corresponding elements of both sides, we get-

$$\left. \begin{array}{lll} u_{11} = a_{11}, & u_{12} = a_{12}, & u_{13} = a_{13} \\ l_{21}u_{11} = a_{21}, & l_{21}u_{12} + u_{22} = a_{22}, & l_{21}u_{13} + u_{23} = a_{23} \\ l_{31}u_{11} = a_{31}, & l_{31}u_{12} + l_{32}u_{22} = a_{32}, & l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33} \end{array} \right\} \dots (2)$$

From the above equations, we can obtain: $l_{21}, l_{31}, l_{32}, u_{11}, u_{12}, u_{13}, u_{22}, u_{23}$ & u_{33}
And thus we can find L and U .

Example 1.5.1: Factorize/Decompose the given matrix A into the LU form.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution: We can write as- $A = L \cdot U$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \dots (1)$$

Multiplying the matrices on the right hand sides of equation... (1) and equating the corresponding elements of both sides, we get-

$$\left. \begin{array}{lll} u_{11} = 2, & u_{12} = 3, & u_{13} = 1 \\ l_{21}u_{11} = 1, & l_{21}u_{12} + u_{22} = 2, & l_{21}u_{13} + u_{23} = 3 \\ l_{31}u_{11} = 3, & l_{31}u_{12} + l_{32}u_{22} = 1, & l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2 \end{array} \right\} \quad \dots (2)$$

$$\Rightarrow \left. \begin{array}{lll} u_{11} = 2, & u_{12} = 3, & u_{13} = 1 \\ l_{21} = 1/2, & u_{22} = 1/2, & u_{23} = 5/2 \\ l_{31} = 3/2, & l_{32} = -7, & u_{33} = 18 \end{array} \right\} \quad \dots (3)$$

It follows that:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

Verification: $R.H.S = L \cdot U$

$$\begin{aligned} &= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \\ &= A = L \cdot H.S \end{aligned}$$

Example 1.5.2: Decompose the given matrix A into the form of LU where L is unit lower triangular and U an upper triangular matrix.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution: We can write as- $A = L \cdot U$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \dots (1)$$

Step-1: Reduce given matrix A into Upper-Triangular matrix U using Gauss-Elimination:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Perform: $R_2 \rightarrow R_2 - \frac{1}{2}R_1$
 $R_3 \rightarrow R_3 - \frac{3}{2}R_1$

$$\frac{1}{2}R_1 = \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{3}{2}R_1 = \begin{bmatrix} 3 & \frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\cong \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & -7/2 & 1/2 \end{bmatrix}$$

Perform: $R_3 \rightarrow R_3 + 7R_2$

$$7R_2 = \begin{bmatrix} 0 & 7/2 & 35/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

Step-2: Obtain Unit-Lower Triangular Matrix L

$$A = L \cdot U$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \quad \dots (2)$$

We get $2l_{21} = 1 \Rightarrow l_{21} = 1/2$

$2l_{31} = 3 \Rightarrow l_{31} = 3/2$

$3l_{31} + \frac{1}{2}l_{32} = 1 \Rightarrow 3\left(\frac{3}{2}\right) + \frac{1}{2}l_{32} = 1 \Rightarrow l_{32} = -7$

Thus, $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \quad \dots (3)$

Example 1.5.3: Decompose the given matrix A into the form of LU where L is unit lower triangular and U an upper triangular matrix.

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Solution: We can write as- $A = L \cdot U$

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \dots (1)$$

Step-1: Reduce given matrix A into Upper-Triangular matrix U using Gauss-Elimination:

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Perform: $R_2 \rightarrow R_2 - \frac{7}{5}R_1$

$$\frac{7}{5}R_1 = \begin{bmatrix} 7 & -\frac{14}{5} & \frac{7}{5} \end{bmatrix}$$

$R_3 \rightarrow R_3 - \frac{3}{5}R_1$

$$\frac{3}{5}R_1 = \begin{bmatrix} 3 & -\frac{6}{5} & \frac{3}{5} \end{bmatrix}$$

$$\cong \begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 41/5 & 17/5 \end{bmatrix}$$

Perform: $R_3 \rightarrow R_3 - \frac{41}{19}R_2$

$$\frac{41}{19}R_2 = \begin{bmatrix} 0 & \frac{41}{5} & -\frac{1312}{95} \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 0 & 1635/95 \end{bmatrix}$$

Step-2: Obtain Unit-Lower Triangular Matrix L

$$A = L \cdot U$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 0 & 1635/95 \end{bmatrix} \quad \dots (2)$$

We get $5l_{21} = 7 \Rightarrow l_{21} = 7/5$

$5l_{31} = 3 \Rightarrow l_{31} = 3/5$

$-2l_{31} + \frac{19}{5}l_{32} = 7 \Rightarrow -2\left(\frac{3}{5}\right) + \frac{19}{5}l_{32} = 7 \Rightarrow l_{32} = \frac{41}{19}$

Thus,
$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 7/5 & 1 & 0 \\ 3/5 & 41/19 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 0 & 1635/95 \end{bmatrix} \quad \dots (3)$$

(d) LU-Factorization Method: Let us consider a linear system of equations:

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \right\} \quad \dots (1)$$

This can be written in the matrix form as follows:

$$AX = B \quad \dots (2)$$

Substitute in this equation

$$A = L \cdot U \quad \dots (3)$$

Equation ... (2) becomes:

$$L \cdot UX = B \quad \dots (4)$$

If we set:

$$UX = Y \quad \dots (5)$$

Equation ... (4) takes the form:

$$L \cdot Y = B \quad \dots (6)$$

Step-1: We first solve Equation ... (6) :

$$L \cdot Y = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \dots (7)$$

This is equivalent to the system:

$$\left. \begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3 \end{aligned} \right\} \quad \dots (8)$$

The system ... (8) can be solved for y_1 , y_2 and y_3 by forward substitution.

Step-2: We now solve Equation ... (5) :

$$UX = Y$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \dots (9)$$

$$\left. \begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{33}x_3 &= y_3 \end{aligned} \right\} \quad \dots (10)$$

The system ... (10) can be solved for x_1 , x_2 and x_3 by backward substitution.

Example 1.5.2: Solve the given linear system of equations using LU-Factorization Method

$$\begin{cases} 2x + 3y + z = 9 \\ x + 2y + 3z = 6 \\ 3x + 1y + 2z = 8 \end{cases}$$

Solution: We have- $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

Step-1: To find the LU-Factorization of matrix A

We can write as- $A = L \cdot U$... (1)

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad \dots (2)$$

Multiplying the matrices on the right hand sides of equation... (2) and equating the corresponding elements of both sides, we get-

$$\left. \begin{aligned} u_{11} &= 2, & u_{12} &= 3, & u_{13} &= 1 \\ l_{21}u_{11} &= 1, & l_{21}u_{12} + u_{22} &= 2, & l_{21}u_{13} + u_{23} &= 3 \\ l_{31}u_{11} &= 3, & l_{31}u_{12} + l_{32}u_{22} &= 1, & l_{31}u_{13} + l_{32}u_{23} + u_{33} &= 2 \end{aligned} \right\} \quad \dots (3)$$

$$\Rightarrow \left. \begin{aligned} u_{11} &= 2, & u_{12} &= 3, & u_{13} &= 1 \\ l_{21} &= 1/2, & u_{22} &= 1/2, & u_{23} &= 5/2 \\ l_{31} &= 3/2, & l_{32} &= -7, & u_{33} &= 18 \end{aligned} \right\} \quad \dots (4)$$

It follows that:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \quad \dots (5)$$

Step-2: To find the solution of given linear system of equations by using LU-Factorization Method

$$L \cdot Y = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \dots (6)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \quad \dots (7)$$

This is equivalent to the system:

$$\left. \begin{aligned} y_1 &= 9 \\ \frac{1}{2}y_1 + y_2 &= 6 \\ \frac{3}{2}y_1 - 7y_2 + y_3 &= 8 \end{aligned} \right\} \quad \dots (8)$$

The system ...(8) can be solved for y_1 , y_2 and y_3 by forward substitution.

$$\Rightarrow \quad y_1 = 9 \quad y_2 = \frac{3}{2} \quad y_3 = 5$$

Finally, we solve Equation:

$$UX = Y$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \dots (9)$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix} \quad \dots (10)$$

$$\left. \begin{aligned} 2x_1 + 3x_2 + x_3 &= 9 \\ \frac{1}{2}x_2 + \frac{5}{2}x_3 &= \frac{3}{2} \\ 18x_3 &= 5 \end{aligned} \right\} \quad \dots (11)$$

The system ...(11) can be solved for x_1 , x_2 and x_3 by backward substitution.

$$\Rightarrow \quad x_1 = \frac{35}{18} \quad x_2 = \frac{29}{18} \quad x_3 = \frac{5}{18}$$

Example 1.5.3: Solve the given linear system of equations using LU-Factorization Method.

$$\left. \begin{array}{l} 5x - 2y + z = 4 \\ 7x + y - 5z = 8 \\ 3x + 7y + 4z = 10 \end{array} \right\}, \text{ Given: } L = \begin{bmatrix} 1 & 0 & 0 \\ 7/5 & 1 & 0 \\ 3/5 & 41/19 & 1 \end{bmatrix} \text{ \& } U = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 0 & 1635/95 \end{bmatrix}$$

Solution: LU-Factorization Method: We start with $L \cdot Y = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \dots (6)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 7/5 & 1 & 0 \\ 3/5 & 41/19 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix} \quad \dots (7)$$

This is equivalent to the system:

$$\left. \begin{array}{l} y_1 = 4 \\ \frac{7}{5}y_1 + y_2 = 8 \\ \frac{3}{5}y_1 + \frac{41}{19}y_2 + y_3 = 10 \end{array} \right\} \quad \dots (8)$$

The system ... (8) can be solved for y_1, y_2 and y_3 by forward substitution.

$$\Rightarrow \quad y_1 = 4 \quad y_2 = \frac{12}{5} \quad y_3 = \frac{46}{19}$$

Finally, we solve Equation: $UX = Y$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \dots (9)$$

$$\Rightarrow \begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 0 & 1635/95 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12/5 \\ 46/19 \end{bmatrix} \quad \dots (10)$$

$$\left. \begin{array}{l} 5x_1 - 2x_2 + x_3 = 4 \\ \frac{19}{5}x_2 - \frac{32}{5}x_3 = \frac{12}{5} \\ \frac{1635}{95}x_3 = \frac{46}{19} \end{array} \right\} \quad \dots (11)$$

The system ... (11) can be solved for x_1, x_2 and x_3 by backward substitution.

$$\Rightarrow \quad x_1 = 1 \frac{13}{109} = \frac{123}{109} \quad x_2 = \frac{284}{327} \quad x_3 = \frac{46}{327}$$

1.8 Gauss-Jacobi Iterative Method:

Let us consider a linear system of equations-

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \end{aligned} \right\} \quad \dots (1)$$

Where, a_{ii} is the largest coefficient in the i^{th} equation so that the coefficient matrix becomes **Diagonally Dominant**. The given linear system of equations can be written as-

$$\left. \begin{aligned} x_1 &= \frac{1}{a_{11}} [b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n] \\ x_2 &= \frac{1}{a_{22}} [b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n] \\ \vdots &\vdots \\ x_n &= \frac{1}{a_{nn}} [b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n(n-1)}x_{n-1}] \end{aligned} \right\} \quad \dots (2)$$

In **Jacobi method** the first approximations to the given linear system of equations are given by-

$$\left. \begin{aligned} x_1^{(1)} &= \frac{1}{a_{11}} [b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \cdots - a_{1n}x_n^{(0)}] \\ x_2^{(1)} &= \frac{1}{a_{22}} [b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)} - \cdots - a_{2n}x_n^{(0)}] \\ \vdots &\vdots \\ x_n^{(1)} &= \frac{1}{a_{nn}} [b_n - a_{n1}x_1^{(0)} - a_{n2}x_2^{(0)} - \cdots - a_{n(n-1)}x_{n-1}^{(0)}] \end{aligned} \right\} \quad \dots (3)$$

In order to get the first approximations, generally, we choose the initial approximations as follows: $x_1^{(0)} = 0, \quad x_2^{(0)} = 0, \quad x_3^{(0)} = 0, \quad \dots \quad x_n^{(0)} = 0$

The second approximations to the given linear system of equations are given by-

$$\left. \begin{aligned} x_1^{(2)} &= \frac{1}{a_{11}} [b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)} - \cdots - a_{1n}x_n^{(1)}] \\ x_2^{(2)} &= \frac{1}{a_{22}} [b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(1)} - \cdots - a_{2n}x_n^{(1)}] \\ \vdots &\vdots \\ x_n^{(2)} &= \frac{1}{a_{nn}} [b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \cdots - a_{n(n-1)}x_{n-1}^{(1)}] \end{aligned} \right\} \quad \dots (4)$$

The second approximations are substituted into the **R.H.S.** of equation (4) to obtain the third approximations and so on. The entire process is repeated until the values of $x_1, x_2, x_3, \dots, x_n$ are obtained to desire accuracy.

Example 1.6.1: Use Gauss-Jacobi's iterative technique to find approximate solution for the following linear system of equations (3 iterations only).

$$\begin{cases} 4x + 11y + z = 33 \\ 8x - 3y + 2z = 20 \\ 6x + 3y + 12z = 35 \end{cases}$$

Answer: Re-writing the given linear system of equations to make the coefficient matrix as Diagonally-Dominant:

$$\Rightarrow \begin{cases} 8x - 3y + 2z = 20 \\ 4x + 11y + z = 33 \\ 6x + 3y + 12z = 35 \end{cases} \quad \dots (1)$$

Thus, the linear system of equations can be written in the form-

$$\begin{cases} x = \frac{1}{8}[20 + 3y - 2z] \\ y = \frac{1}{11}[33 - 4x - z] \\ z = \frac{1}{12}[35 - 6x - 3y] \end{cases} \quad \dots (2)$$

Gauss-Jacobi's First Approximation:

$$\begin{cases} x^{(1)} = \frac{1}{8}[20 + 3y^{(0)} - 2z^{(0)}] \\ y^{(1)} = \frac{1}{11}[33 - 4x^{(0)} - z^{(0)}] \\ z^{(1)} = \frac{1}{12}[35 - 6x^{(0)} - 3y^{(0)}] \end{cases} \quad \dots (3)$$

Starting with the initial approximation: $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$

$$\begin{cases} x^{(1)} = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5 \\ y^{(1)} = \frac{1}{11}[33 - 4(0) - (0)] = 3.0 \\ z^{(1)} = \frac{1}{12}[35 - 6(0) - 3(0)] = 2.9166667 \end{cases} \quad \dots (4)$$

Gauss-Jacobi's Second Approximation:

$$\begin{cases} x^{(2)} = \frac{1}{8}[20 + 3y^{(1)} - 2z^{(1)}] = 2.8958330 \\ y^{(2)} = \frac{1}{11}[33 - 4x^{(1)} - z^{(1)}] = 2.3560606 \\ z^{(2)} = \frac{1}{12}[35 - 6x^{(1)} - 3y^{(1)}] = 0.9166666 \end{cases} \quad \dots (5)$$

Gauss-Jacobi's Third Approximation:

$$\left. \begin{aligned} x^{(3)} &= \frac{1}{8} [20 + 3y^{(2)} - 2z^{(2)}] = 3.1543561 \\ y^{(3)} &= \frac{1}{11} [33 - 4x^{(2)} - z^{(2)}] = 2.0303300 \\ z^{(3)} &= \frac{1}{12} [35 - 6x^{(2)} - 3y^{(2)}] = 0.8797348 \end{aligned} \right\} \dots (6)$$

Proceeding in this way, after 12th approximation, we find that the two successive approximations are same up to four decimal places.

Hence at this stage, we get:

$$x = 3.0167, \quad y = 1.9858, \quad z = 0.9118$$

Example 1.6.2: Use **Gauss-Jacobi Iterative Technique** to find approximate solution to the following linear system of equations. (Perform 3 iterations only):

$$\left. \begin{array}{l} x + y + 54z = 110 \\ 6x + 15y + 2z = 72 \\ 27x + 6y - z = 85 \end{array} \right\} \begin{array}{l} \text{Diagonally} \\ \text{Dominant} \end{array} \Rightarrow \left. \begin{array}{l} 27x + 6y - z = 85 \\ 6x + 15y + 2z = 72 \\ x + y + 54z = 110 \end{array} \right\}$$

Solution: We observe that the coefficient matrix has become as **Diagonally-Dominant**.

Jacobi First Iteration: Starting with $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$, we get-

$$\left. \begin{array}{l} x^{(1)} = \frac{1}{27} [85 - 6y^{(0)} + z^{(0)}] = 3.148148148 \\ y^{(1)} = \frac{1}{15} [72 - 6x^{(0)} - 2z^{(0)}] = 4.8 \\ z^{(1)} = \frac{1}{54} [110 - x^{(0)} - y^{(0)}] = 2.037037037 \end{array} \right\}$$

Jacobi Second Iteration:

$$\left. \begin{array}{l} x^{(2)} = \frac{1}{27} [85 - 6y^{(1)} + z^{(1)}] = 2.156927298 \\ y^{(2)} = \frac{1}{15} [72 - 6x^{(1)} - 2z^{(1)}] = 3.269135803 \\ z^{(2)} = \frac{1}{54} [110 - x^{(1)} - y^{(1)}] = 1.889849108 \end{array} \right\}$$

Jacobi Third Iteration:

$$\left. \begin{array}{l} x^{(3)} = \frac{1}{27} [85 - 6y^{(2)} + z^{(2)}] = 2.491667937 \\ y^{(3)} = \frac{1}{15} [72 - 6x^{(2)} - 2z^{(2)}] = 3.6852492 \\ z^{(3)} = \frac{1}{54} [110 - x^{(2)} - y^{(2)}] = 1.936554387 \end{array} \right\}$$

We can write all the values in the table as:

Jacobi Iteration Technique			
K	$x^{(K+1)}$	$y^{(K+1)}$	$z^{(K+1)}$
0	$x^{(1)} = 3.148148148$	$y^{(1)} = 4.8$	$z^{(1)} = 2.037037037$
1	$x^{(2)} = 2.156927298$	$y^{(2)} = 3.269135803$	$z^{(2)} = 1.889849108$
2	$x^{(3)} = 2.491667937$	$y^{(3)} = 3.6852492$	$z^{(3)} = 1.936554387$

Example 1.6.3: Solve the given linear system of equations using Gauss-Jacobi Method up to 3 iterations. (Starting with: $x^{(0)} = 0.5$, $y^{(0)} = 0.5$, $z^{(0)} = 0.5$)

$$\left. \begin{aligned} 6x - 3y + z &= 11 \\ -x + 7y - z &= -10 \\ -2x - y + 8z &= 15 \end{aligned} \right\}$$

Solution: The coefficient matrix of the given linear system of equations is already in Diagonally Dominant:

$$\left. \begin{aligned} 6x - 3y + z &= 11 \\ -x + 7y - z &= -10 \\ -2x - y + 8z &= 15 \end{aligned} \right\}$$

Gauss-Jacobi First Iteration: Starting with $x^{(0)} = 0.5$, $y^{(0)} = 0.5$, $z^{(0)} = 0.5$, we get-

$$\left. \begin{aligned} x^{(1)} &= \frac{1}{6} [11 + 3y^{(0)} - z^{(0)}] = 2.000000000 \\ y^{(1)} &= \frac{1}{7} [-10 + x^{(0)} + z^{(0)}] = -1.285714286 \\ z^{(1)} &= \frac{1}{8} [15 + 2x^{(0)} + y^{(0)}] = 2.062500000 \end{aligned} \right\}$$

Gauss- Jacobi Second Iteration:

$$\left. \begin{aligned} x^{(2)} &= \frac{1}{6} [11 + 3y^{(1)} - z^{(1)}] = 0.846726190 \\ y^{(2)} &= \frac{1}{7} [-10 + x^{(1)} + z^{(1)}] = -0.848214286 \\ z^{(2)} &= \frac{1}{8} [15 + 2x^{(1)} + y^{(1)}] = 2.214285714 \end{aligned} \right\}$$

Gauss- Jacobi Third Iteration:

$$\left. \begin{aligned} x^{(3)} &= \frac{1}{6} [11 + 3y^{(2)} - z^{(2)}] = 1.040178571 \\ y^{(3)} &= \frac{1}{7} [-10 + x^{(2)} + z^{(2)}] = -0.991284014 \\ z^{(3)} &= \frac{1}{8} [15 + 2x^{(2)} + y^{(2)}] = 1.980654762 \end{aligned} \right\}$$

We can write all the values in the table as:

Gauss-Jacobi Iteration Technique			
K	$x^{(K+1)}$	$y^{(K+1)}$	$z^{(K+1)}$
0	$x^{(1)} = 2.000000000$	$y^{(1)} = -1.285714286$	$z^{(1)} = 2.062500000$
1	$x^{(2)} = 0.846726190$	$y^{(2)} = -0.848214286$	$z^{(2)} = 2.214285714$
2	$x^{(3)} = 1.040178571$	$y^{(3)} = -0.991284014$	$z^{(3)} = 1.980654762$

1.9 The Gauss-Seidel Method:

This method is a modified form of **Gauss-Jacobi Iterative Method**. In Jacobi Iterative method, no element of $x_i^{(k+1)}$ replaces $x_i^{(k)}$ entirely for the next cycle of computation. However, in the Gauss-Seidel method, the corresponding element of $x_i^{(k+1)}$ replaces those of $x_i^{(k)}$ as soon as they become available.

Let us consider a linear system of equations-

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \end{aligned} \right\} \quad \dots (1)$$

Where, a_{ii} is the largest coefficient in the i^{th} equation so that the coefficient matrix becomes **Diagonally Dominant**. The given linear system of equations can be written as-

$$\left. \begin{aligned} x_1 &= \frac{1}{a_{11}} [b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n] \\ x_2 &= \frac{1}{a_{22}} [b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n] \\ \vdots &\vdots \\ x_n &= \frac{1}{a_{nn}} [b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n(n-1)}x_{n-1}] \end{aligned} \right\} \quad \dots (2)$$

In **Gauss-Seidel method** the first approximations to the given linear system of equations are given by-

$$\left. \begin{aligned} x_1^{(1)} &= \frac{1}{a_{11}} [b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \cdots - a_{1n}x_n^{(0)}] \\ x_2^{(1)} &= \frac{1}{a_{22}} [b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)} - \cdots - a_{2n}x_n^{(0)}] \\ \vdots &\vdots \\ x_n^{(1)} &= \frac{1}{a_{nn}} [b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \cdots - a_{n(n-1)}x_{n-1}^{(1)}] \end{aligned} \right\} \quad \dots (3)$$

In order to get the first approximations, generally, we choose the initial approximations as follows: $x_1^{(0)} = 0, \quad x_2^{(0)} = 0, \quad x_3^{(0)} = 0, \quad \dots \quad x_n^{(0)} = 0$ (If not given)

The second approximations to the given linear system of equations are given by-

$$\left. \begin{aligned} x_1^{(2)} &= \frac{1}{a_{11}} [b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)} - \cdots - a_{1n}x_n^{(1)}] \\ x_2^{(2)} &= \frac{1}{a_{22}} [b_2 - a_{21}x_1^{(2)} - a_{23}x_3^{(0)} - \cdots - a_{2n}x_n^{(0)}] \\ \vdots &\vdots \\ x_n^{(2)} &= \frac{1}{a_{nn}} [b_n - a_{n1}x_1^{(2)} - a_{n2}x_2^{(2)} - \cdots - a_{n(n-1)}x_{n-1}^{(2)}] \end{aligned} \right\} \quad \dots (4)$$

In general, the $(k + 1)^{th}$ approximation is computed as follows-

$$\left. \begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{11}} \left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \dots - a_{1n}x_n^{(k)} \right] \\ x_2^{(k+1)} &= \frac{1}{a_{22}} \left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \dots - a_{2n}x_n^{(k)} \right] \\ &\vdots \\ x_n^{(k+1)} &= \frac{1}{a_{nn}} \left[b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \dots - a_{n(n-1)}x_{n-1}^{(k+1)} \right] \end{aligned} \right\} \dots (5)$$

Note: The rate of convergence of Gauss-Seidel method is approximately twice that of Gauss-Jacobi Iterative method.

Example 1.7.1: Use Gauss-Seidel iterative technique to find approximate solution for the following linear system of equations-

$$\left. \begin{aligned} 10x_1 - x_2 + 2x_3 &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned} \right\} \dots (1)$$

Starting with the initial approximation: $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$, $x_4^{(0)} = 0$, and iterating until $\frac{\|x^{(k+1)} - x^{(k)}\|_{\infty}}{\|x^{(k+1)}\|_{\infty}} < 10^{-3}$

Solution: We observe that the coefficient matrix of the given linear system of equations is already of **Diagonally-Dominant**.

Thus, the given linear system of equations can be written in the form-

$$\left. \begin{aligned} x_1 &= \frac{1}{10} [6 + x_2 - 2x_3 - 0x_4] \\ x_2 &= \frac{1}{11} [25 + x_1 + x_3 - 3x_4] \\ x_3 &= \frac{1}{10} [-11 - 2x_1 + x_2 + x_4] \\ x_4 &= \frac{1}{8} [15 - 0x_1 - 3x_2 + x_3] \end{aligned} \right\} \dots (2)$$

Gauss-Seidel First Approximation: Starting with the initial approximation:

$$x_2^{(0)} = 0, \quad x_3^{(0)} = 0, \quad x_4^{(0)} = 0$$

$$\left. \begin{aligned} x_1^{(1)} &= \frac{1}{10} [6 + x_2^{(0)} - 2x_3^{(0)} - 0x_4^{(0)}] = 0.6000 \\ x_2^{(1)} &= \frac{1}{11} [25 + x_1^{(1)} + x_3^{(0)} - 3x_4^{(0)}] = 2.3272 \\ x_3^{(1)} &= \frac{1}{10} [-11 - 2x_1^{(1)} + x_2^{(1)} + x_4^{(0)}] = -0.9873 \\ x_4^{(1)} &= \frac{1}{8} [15 - 0x_1^{(1)} - 3x_2^{(1)} + x_3^{(1)}] = 0.8789 \end{aligned} \right\} \dots (3)$$

Gauss-Seidel Second Approximation:

$$\left. \begin{aligned} x_1^{(2)} &= \frac{1}{10} [6 + x_2^{(1)} - 2x_3^{(1)} - 0x_4^{(1)}] = 1.0300 \\ x_2^{(2)} &= \frac{1}{11} [25 + x_1^{(2)} + x_3^{(1)} - 3x_4^{(1)}] = 2.0370 \\ x_3^{(2)} &= \frac{1}{10} [-11 - 2x_1^{(2)} + x_2^{(2)} + x_4^{(1)}] = -1.0140 \\ x_4^{(2)} &= \frac{1}{8} [15 - 0x_1^{(2)} - 3x_2^{(2)} + x_3^{(2)}] = 0.9844 \end{aligned} \right\} \dots (4)$$

Gauss-Seidel Third Approximation:

$$\left. \begin{aligned} x_1^{(3)} &= \frac{1}{10} [6 + x_2^{(2)} - 2x_3^{(2)} - 0x_4^{(2)}] = 1.0065 \\ x_2^{(3)} &= \frac{1}{11} [25 + x_1^{(3)} + x_3^{(2)} - 3x_4^{(2)}] = 2.0036 \\ x_3^{(3)} &= \frac{1}{10} [-11 - 2x_1^{(3)} + x_2^{(3)} + x_4^{(2)}] = -1.0025 \\ x_4^{(3)} &= \frac{1}{8} [15 - 0x_1^{(3)} - 3x_2^{(3)} + x_3^{(3)}] = 0.9983 \end{aligned} \right\} \dots (5)$$

Gauss-Seidel Fourth Approximation:

$$\left. \begin{aligned} x_1^{(4)} &= \frac{1}{10} [6 + x_2^{(3)} - 2x_3^{(3)} - 0x_4^{(3)}] = 1.0009 \\ x_2^{(4)} &= \frac{1}{11} [25 + x_1^{(4)} + x_3^{(3)} - 3x_4^{(3)}] = 2.0003 \\ x_3^{(4)} &= \frac{1}{10} [-11 - 2x_1^{(4)} + x_2^{(4)} + x_4^{(3)}] = -1.0003 \\ x_4^{(4)} &= \frac{1}{8} [15 - 0x_1^{(4)} - 3x_2^{(4)} + x_3^{(4)}] = 0.9999 \end{aligned} \right\} \dots (6)$$

Gauss-Seidel Fifth Approximation:

$$\left. \begin{aligned} x_1^{(5)} &= \frac{1}{10} [6 + x_2^{(4)} - 2x_3^{(4)} - 0x_4^{(4)}] = 1.0001 \\ x_2^{(5)} &= \frac{1}{11} [25 + x_1^{(5)} + x_3^{(4)} - 3x_4^{(4)}] = 2.0000 \\ x_3^{(5)} &= \frac{1}{10} [-11 - 2x_1^{(5)} + x_2^{(5)} + x_4^{(4)}] = -1.0000 \\ x_4^{(5)} &= \frac{1}{8} [15 - 0x_1^{(5)} - 3x_2^{(5)} + x_3^{(5)}] = 1.0000 \end{aligned} \right\} \dots (7)$$

Test for convergence: We know that for $k = 0, 1, 2, 3, 4, \dots, n$ iterations-

$$\|x^{(k+1)}\|_{\infty} = \text{Max of } |x^{(k+1)}| \quad \text{and} \quad \|x^{(k+1)} - x^{(k)}\|_{\infty} = \text{Max of } |x^{(k+1)} - x^{(k)}|$$

Thus for $k = 4$

$$\|x^{(5)}\|_{\infty} = \text{Max of } |x^{(5)}| = 2.0000$$

and

$$\|x^{(5)} - x^{(4)}\|_{\infty} = \text{Max of } |x^{(5)} - x^{(4)}| = 0.0008 = 8.0 \times 10^{-4}$$

Thus

$$\frac{\|x^{(5)} - x^{(4)}\|_{\infty}}{\|x^{(5)}\|_{\infty}} = \frac{8.0 \times 10^{-4}}{2.0000} = 4 \times 10^{-4} < 10^{-3}$$

We stop after 5th iteration because of the desired accuracy.

Example 1.7.2: Use **Gauss-Seidel iterative technique** to find approximate solutions to the following linear system of equations. (Perform 3 iterations only)

$$\begin{cases} 3u + 8v + 29w = 71 \\ 7u + 52v + 13w = 104 \\ 83u + 11v - 4w = 95 \end{cases}$$

Answer: Re-writing the given linear system of equations to make coefficient matrix as **Diagonally-Dominant**:

$$\begin{cases} 83u + 11v - 4w = 95 \\ 7u + 52v + 13w = 104 \\ 3u + 8v + 29w = 71 \end{cases}$$

Gauss-Seidel First Iteration: Starting with $v^{(0)} = 0$, $w^{(0)} = 0$, we get-

$$\begin{cases} u^{(1)} = \frac{1}{83} [95 - 11v^{(0)} + 4w^{(0)}] = 1.144578313 \\ v^{(1)} = \frac{1}{52} [104 - 7u^{(1)} - 13w^{(0)}] = 1.84592215 \\ w^{(1)} = \frac{1}{29} [71 - 3u^{(1)} - 8v^{(1)}] = 1.820651306 \end{cases}$$

Gauss-Seidel Second Iteration:

$$\begin{cases} u^{(2)} = \frac{1}{83} [95 - 11v^{(1)} + 4w^{(1)}] = 0.9876802599 \\ v^{(2)} = \frac{1}{52} [104 - 7u^{(2)} - 13w^{(1)}] = 1.411880215 \\ w^{(2)} = \frac{1}{29} [71 - 3u^{(2)} - 8v^{(2)}] = 1.956617845 \end{cases}$$

Gauss-Seidel Third Iteration:

$$\begin{cases} u^{(3)} = \frac{1}{83} [95 - 11v^{(2)} + 4w^{(2)}] = 1.051756494 \\ v^{(3)} = \frac{1}{52} [104 - 7u^{(3)} - 13w^{(2)}] = 1.369262934 \\ w^{(3)} = \frac{1}{29} [71 - 3u^{(3)} - 8v^{(3)}] = 1.96174576 \end{cases}$$

We can write all the values in the table as:

Gauss-Seidel Iteration Technique			
K	$u^{(K+1)}$	$v^{(K+1)}$	$w^{(K+1)}$
0	$u^{(1)} = 1.144578313$	$v^{(1)} = 1.84592215$	$w^{(1)} = 1.820651306$
1	$u^{(2)} = 0.9876802599$	$v^{(2)} = 1.411880215$	$w^{(2)} = 1.956617845$
2	$u^{(3)} = 1.051756494$	$v^{(3)} = 1.369262934$	$w^{(3)} = 1.96174576$

Exercise 1.1: Numerical Solution of Linear System of Equations

1. Choose the correct answer (Questions 1.1 -1.4)

- 1.1 When **Gauss-Jordan method** is used to solve the matrix equation $AX = B$, then the coefficient matrix A is transformed to the:
- (a) Upper triangular matrix
 - (b) Diagonal matrix
 - (c) Zero matrix
 - (d) None of these
- 1.2 Which of the following **numerical methods** provide with the **exact solutions** for a given **linear system of equations**:
- (a) Gauss-Jacobi method
 - (b) Gauss-Seidel method
 - (c) Gauss-Elimination method and Gauss-Jordan method
 - (d) None of these
- 1.3 Using **back substitution method** find the solution of given **linear system of equations**:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 5 \end{bmatrix}$$

- (a) $x = 7, y = -9, z = 5$
 - (b) $x = -7, y = 9, z = 5$
 - (c) $x = -9, y = 7, z = 5$
 - (d) None of these
- 1.4 The solution of a linear system of equations: $4x + y + 3z = 17$, $x + 5y + z = 14$, $2x - y + 8z = 12$ by **Gauss Jacobi Method** for first iteration (Starting with $x^{(0)} = 1$, $y^{(0)} = 1$ & $z^{(0)} = 1$) is
- (a) $x=4.25, y=2.8, z=1.5$
 - (b) $x=4, y=2.0, z = 1.0$
 - (c) $x= 4.25, y=1.66, z=1.1$
 - (d) $x= 3.25, y= 2.40, z = 1.375$
2. Use **Gauss Elimination** with backward substitution to solve the following **linear system of equations**:

$$\begin{aligned} x + 2y + z &= 8 \\ 2x + 3y + 4z &= 20 \\ 4x + 3y + 2z &= 16 \end{aligned}$$

3. Use **Gauss Elimination** with backward substitution to solve the following **linear system of equations**:

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8 \\ 2x_1 + 5x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 + 4x_3 &= 11 \end{aligned}$$

4. Use **Gauss-Jordan Method** to solve the following **linear system of equations**:

$$\begin{aligned} x + 2y + z &= 8 \\ 2x + 3y + 4z &= 20 \\ 4x + 3y + 2z &= 16 \end{aligned}$$

5. Use **Gauss-Jordan Method** to solve the following **linear system of equations**:

$$\begin{aligned} 4x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \end{aligned}$$

6. Decompose the given matrix **A** into the form of **LU** where **L** is unit lower triangular and **U** an upper triangular matrix.

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

7. Decompose the given matrix **A** into the form of **LU** where **L** is unit lower triangular and **U** an upper triangular matrix.

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

8. Find the first **three iterations** of the **Gauss-Jacobi Method** for the following **linear system of equations** using $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$

$$(a) \quad \left. \begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{aligned} \right\}$$

$$(b) \quad \left. \begin{aligned} 4x_1 + x_2 - x_3 &= 5 \\ -x_1 + 3x_2 + x_3 &= -4 \\ 2x_1 + 2x_2 + 5x_3 &= 1 \end{aligned} \right\}$$

9. Use **Gauss-Jacobi Method** to find approximate solutions to the following linear system of equations-

$$\left. \begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9 \end{aligned} \right\}$$

10. Find the first three iterations of the **Gauss-Seidel Method** for the following linear system of equations using $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$

$$(a) \left. \begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{aligned} \right\}$$

$$(b) \left. \begin{aligned} 4x_1 + x_2 - x_3 &= 5 \\ -x_1 + 3x_2 + x_3 &= -4 \\ 2x_1 + 2x_2 + 5x_3 &= 1 \end{aligned} \right\}$$

11. Use **Gauss-Seidel iterative technique** to find approximate solutions to the following linear system of equations-

$$\left. \begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9 \end{aligned} \right\}$$

Numerical Analysis-1

(For Science College Students)

MATH-316

Chapter-2: Numerical Solution of Nonlinear Equations

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Chapter-2: Numerical Solution of Nonlinear Equations

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2.1 Nonlinear Equations:

The general form of a nonlinear equation is $f(x) = 0$ where f a nonlinear function of the variable is x .

e.g.:

$$x^4 + x^3 + 1 = 0$$

$$xe^{-x} - 7 = 0$$

$$\log x - x = 0$$

$$x^2 + \ln(x - 3) + e^x \sin x = 0$$

2.2 Roots of an Equations $f(x) = 0$:

In order to find the solution of an equation $f(x) = 0$, we find those values of x for which $f(x) = 0$ is satisfied, such values of x are called the roots of $f(x) = 0$

Thus, $x = a$ is a root of an equation: $f(x) = 0 \Leftrightarrow f(a) = 0$

A root of the equation $f(x) = 0$ is also called the **zero** of the function $f(x)$

In this chapter, we shall describe some numerical methods to approximate the solutions of equations of the type $f(x) = 0$ to an arbitrarily high accuracy.

2.3 The Bisection Method:

This method is due to Bolzano and based on the repeated application of intermediate value property.

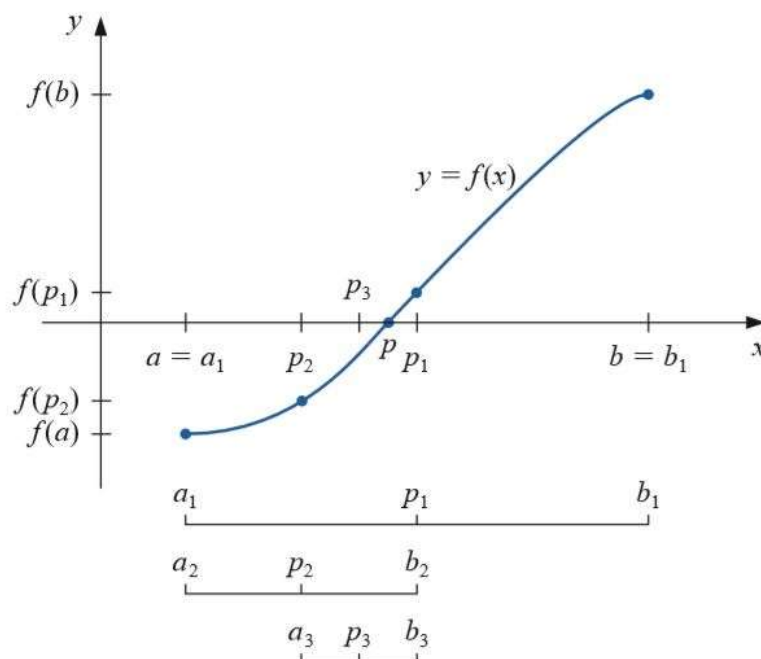
Suppose $f(x)$ is a continuous function defined on the interval $[a, b]$ and we wish to locate the root of an equation:

$$f(x) = 0 \quad \text{in the interval} \quad [a, b].$$

Let- $f(a) \cdot f(b) < 0 \Leftrightarrow f(a)$ and $f(b)$ are of opposite sign.

Then, the intermediate value theorem implies that there exist at least one root p in (a, b) such that

$$f(p) = 0$$



Let- $f(a) < 0$ and $f(b) > 0$ then approximate value of root (p_1) is given by: $p_1 = \frac{a+b}{2}$

- If $f(p_1) = 0$ then $p = p_1$ is correct root and we are done.
- If $f(p_1) \neq 0$ then the root either lies in $(a, \frac{a+b}{2})$ or $(\frac{a+b}{2}, b)$ depending on whether $f(p_1)$ is negative or positive.
- We again bisect the interval and repeat the process until the root is obtained of desired accuracy.

Working Rule: Suppose $f(x)$ is a continuous function defined on the interval $[a, b]$

1. To begin, set $a_1 = a$ and $b_1 = b$ then 1st approximation, $p_1 = \frac{a_1+b_1}{2}$
2. If $f(p_1) \neq 0$ and
 - (a) If $f(p_1)$ and $f(a_1)$ have the same sign then $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$
 - (b) If $f(p_1)$ and $f(a_1)$ have the opposite sign then $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$
3. Repeat the process to the interval $[a_2, b_2]$ and so on.

Example 2.4.1: Show that $f(x) = x^3 + 4x^2 - 10$ has a root in $(1, 2)$ and use **Bisection Method** to determine an approximation to the root correct to 3 decimal places.

Solution: We have $f(x) = x^3 + 4x^2 - 10$ $a_1 = 1$ and $a_2 = 2$

Since, $f(1) = (1)^3 + 4(1)^2 - 10 = -5 < 0$ (**Negative**)

and $f(2) = (2)^3 + 4(2)^2 - 10 = 14 > 0$ (**Positive**)

Therefore, by intermediate theorem, the function $f(x)$ has a root in $[1, 2]$.

Iteration Table:

Bisection Method				
No of Approximation n	a_n (-)	b_n (+)	$p_n = \frac{a_n + b_n}{2}$	$f(p_n) = (p_n)^3 + 4(p_n)^2 - 10$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	-1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	-0.84839
5	1.3125	1.375	1.34375	-0.35098
6	1.34375	1.375	1.359375	-0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	-0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	-0.01605
11	1.364257813	1.365234375	1.364746094	-0.00799

We observe that p_{10} and p_{11} are same up to **three** decimal places. Thus, $p = 1.364$ is correct root of the function $f(x)$ to **three** decimal places.

Example 2.4.2: Use the **Bisection Method** to approximate a solution of the equation: $\sin 10x + \cos 3x = 0$ on the interval $[4.71, 4.715]$ with an accuracy of 10^{-3}

Solution: Let $f(x) = \sin(10x) + \cos(3x)$

Since, $f(4.71) = \sin(10(4.71)) + \cos(3(4.71)) = 0.0310544113 > 0$ (**Positive**)

and $f(4.715) = \sin(10(4.715)) + \cos(3(4.715)) = -0.03394020 < 0$ (**Negative**)

Therefore, by intermediate theorem, the function $f(x)$ has a root in $(4.71, 4.715)$.

Bisection Method					
(n)	a_n (+)	b_n (-)	Approximation $p_n = \frac{a_n + b_n}{2}$	$f(p_n) = \sin(10p_n) + \cos(3p_n)$	$E_R = \frac{ p_{new} - p_{old} }{ p_{new} }$
1	4.71	4.715	$p_1 = 4.7125$	-ive	
2	4.71	4.7125	$p_2 = 4.71125$	+ive	$E_R = \frac{ p_2 - p_1 }{ p_2 } =$
3	4.71125	4.7125	$p_3 = 4.711875$	+ive	$E_R = \frac{ p_3 - p_2 }{ p_3 } =$
4	4.711875	4.7125	$p_4 = 4.7121875$	+ive	$E_R = \frac{ p_4 - p_3 }{ p_4 } =$
5	4.7121875	4.7125	$p_5 = 4.71234375$	+ive	$E_R = \frac{ p_5 - p_4 }{ p_5 } =$
6	4.71234375	4.7125	$p_6 = 4.712421875$	STOP	$E_R = \frac{ p_6 - p_5 }{ p_6 } =$

Since p_4, p_5 and p_6 are identical up to 3 decimal places, therefore $p = 4.712$ is correct with an accuracy of 10^{-3} .

Example 2.4.3: Apply **Bisection Method** to approximate $\sqrt[3]{786}$ in the **interval** $[9.22, 9.23]$, accurate to within 10^{-3} .

Solution: We wish to find:

$$x = \sqrt[3]{786}$$

From Calculator:

$$\sqrt[3]{786} = 9.228706804$$

$$\Rightarrow x^3 = 786$$

$$\Rightarrow x^3 - 786 = 0$$

Let $f(x) = x^3 - 786$

Choose Interval: $[9.22, 9.23]$

Since, $f(9.22) = (9.22)^3 - 786 = -2.222552 < 0$ (**Negative**)

and $f(9.23) = (9.23)^3 - 786 = +0.330467 > 0$ (**Positive**)

Therefore, by intermediate value theorem, the function $f(x)$ must have one root between: 9.22 and 9.23.

Bisection Method					
(n)	a_n (-)	b_n (+)	Approximation $p_n = \frac{a_n + b_n}{2}$	$f(p_n)$ $=$ $(p_n)^3 - 786$	$E_R = \frac{ p_{new} - p_{old} }{ p_{new} }$
1	9.22	9.23	$p_1 = 9.225$	-ive	
2	9.225	9.23	$p_2 = 9.2275$	-ive	$E_R = \frac{ p_2 - p_1 }{ p_2 } =$
3	9.2275	9.23	$p_3 = 9.22875$	+ive	$E_R = \frac{ p_3 - p_2 }{ p_3 } =$
4	9.2275	9.22875	$p_4 = 9.228125$	-ive	$E_R = \frac{ p_4 - p_3 }{ p_4 } =$
5	9.228125	9.22875	$p_5 = 9.2284375$	STOP	$E_R = \frac{ p_5 - p_4 }{ p_5 } =$

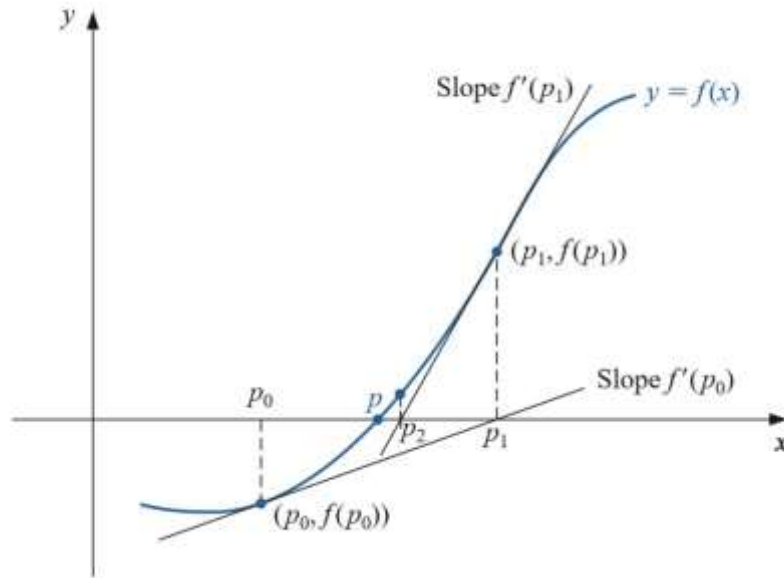
Since p_3, p_4 and p_5 are identical up to 3 decimal places, therefore $p = 9.228$ is correct with an accuracy of 10^{-3} .

2.4 Newton- Raphson Method:

Newton–Raphson Method is one of the most powerful and well known numerical methods for solving a root of the nonlinear equation $f(x) = 0$.

Suppose p denote the exact root of the equation $f(x) = 0$ provided $f'(p) \neq 0$, i.e.

$$f(p) = 0 \quad \dots (1)$$



Let $p_0 \in [a, b]$ be the initial approximation and h be the correction required in p_0 i.e.

$$\begin{aligned} p_0 + h &= p \\ \Rightarrow f(p_0 + h) &= f(p) = 0 \end{aligned} \quad \dots (2)$$

Now, Expanding: $f(p_0 + h)$ by Taylor's theorem-

$$f(p_0 + h) = f(p_0) + hf'(p_0) + \frac{h^2}{2!}f''(p_0) + \dots + \dots \quad \dots (3)$$

Since, h is very small, neglecting second and higher order terms-

$$\begin{aligned} 0 &= f(p_0) + hf'(p_0) \\ \Rightarrow h &= -\frac{f(p_0)}{f'(p_0)} \quad [\text{provided, } f'(p_0) \neq 0] \end{aligned}$$

Substitute the value of h in equation ...(2), we get a better approximation to the root-

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} \approx p$$

Similarly, the improved root over the previous one can be obtained by iteration-

$$\begin{aligned} p_2 &= p_1 - \frac{f(p_1)}{f'(p_1)} \quad [\text{provided, } f'(p_1) \neq 0] \\ p_3 &= p_2 - \frac{f(p_2)}{f'(p_2)} \quad [\text{provided, } f'(p_2) \neq 0] \\ &\vdots \quad \vdots \quad \vdots \\ p_n &= p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad \text{for all } n \geq 1 \quad [\text{provided, } f'(p_{n-1}) \neq 0] \end{aligned}$$

Example 2.5.1: Find the root of $f(x) = x^4 - x - 10$ using **Newton-Raphson Method** that lies near $p_0 = 2$.

Solution: We have $f(x) = x^4 - x - 10$ $p_0 = 2$

$$\Rightarrow f'(x) = 4x^3 - 1$$

$$\Rightarrow f(2) = 2^4 - 2 - 10 = 4$$

$$\Rightarrow f'(2) = 4(2^3) - 1 = 31$$

Applying Newton – Raphson Method: **For $n = 1$**

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$$

$$\Rightarrow p_1 = 2 - \frac{f(2)}{f'(2)} = 1.870967742$$

Example 2.5.2: Use Newton-Raphson Method to find the solution of $f(x) = \cos x - x = 0$, with an accuracy of 10^{-5} , starting with $p_0 = \frac{\pi}{4}$

Solution: We have

$$f(x) = \cos x - x \Rightarrow f(p_{n-1}) = \cos p_{n-1} - p_{n-1} \quad \dots (1)$$

$$\Rightarrow f'(x) = -\sin x - 1 \Rightarrow f'(p_{n-1}) = -\sin p_{n-1} - 1 \quad \dots (2)$$

Applying Newton – Raphson Method-

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad \text{for all } n \geq 1$$

From Equations (1) and (2), we can write as follows-

$$\Rightarrow p_n = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1} \quad \text{for all } n \geq 1$$

For $n = 1$

$$\Rightarrow p_1 = p_0 - \frac{\cos p_0 - p_0}{-\sin p_0 - 1}$$

$$\Rightarrow p_1 = 0.7853981635 - \frac{\cos 0.7853981635 - 0.7853981635}{-\sin 0.7853981635 - 1} = 0.7395361337$$

For $n = 2$

$$\Rightarrow p_2 = p_1 - \frac{\cos p_1 - p_1}{-\sin p_1 - 1}$$

$$\Rightarrow p_2 = 0.7395361337 - \frac{\cos 0.7395361337 - 0.7395361337}{-\sin 0.7395361337 - 1} = 0.7390851781$$

Similarly, we can find the following

$$p_3 = 0.7390851332 \quad \text{and} \quad p_4 = 0.7390851332$$

Writing all in a Table: Starting with, $p_0 = \frac{\pi}{4} = 0.7853981635$

Newton- Raphson Method		
Number of Iteration (n)	Approximation: $p_n = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}$	Percentage Error: $E_p = \frac{ p_{new} - p_{old} }{ p_{new} } \cdot 100$
1	$p_1 = 0.7395361337$	$E_p = \frac{ p_1 - p_0 }{ p_1 } \cdot 100 =$
2	$p_2 = 0.7390851781$	$E_p = \frac{ p_2 - p_1 }{ p_2 } \cdot 100 =$
3	$p_3 = 0.7390851332$	$E_p = \frac{ p_3 - p_2 }{ p_3 } \cdot 100 =$
4	$p_4 = 0.7390851332$	$E_p = \frac{ p_4 - p_3 }{ p_4 } \cdot 100 =$

We observe that p_3 and p_4 are same up to Ten decimal places. Thus, $p = 0.7390851332$ is correct root of the function $f(x)$ up to **Ten places of decimals**.

Example 2.5.3: Use **Newton-Raphson Method** to find solutions accurate to within 10^{-5} to the given problem:

$$\sin x - e^{-x} = 0, \quad \text{for} \quad 6 \leq x \leq 7$$

Solution: We have $f(x) = \sin x - e^{-x}$

Since, $f(6) = \sin 6 - e^{-6} = -0.2818942504 < 0$ (**Negative**)

and $f(7) = \sin 7 - e^{-7} = +0.6560747168 > 0$ (**Positive**)

Therefore, by intermediate theorem, the function $f(x)$ has at least one real root between 6 and 7.

Now,

$$f(x) = \sin x - e^{-x} \Rightarrow f(p_{n-1}) = \sin(p_{n-1}) - e^{-(p_{n-1})} \quad \dots (1)$$

$$\Rightarrow f'(x) = \cos x + e^{-x} \Rightarrow f'(p_{n-1}) = \cos(p_{n-1}) + e^{-(p_{n-1})} \quad \dots (2)$$

Applying Newton-Raphson Method-

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad \text{for all } n \geq 1$$

From Equations (1) and (2), we can write -

$$\Rightarrow p_n = p_{n-1} - \left[\frac{\sin(p_{n-1}) - e^{-(p_{n-1})}}{\cos(p_{n-1}) + e^{-(p_{n-1})}} \right] \quad \text{for all } n \geq 1$$

Starting with, $p_0 = 6$, (Because $f(6) = -0.28$ is near to Zero in comparison of $f(7) = +0.65$).

$$\begin{aligned} \text{For } n = 1 & \Rightarrow p_1 = p_0 - \left[\frac{\sin(p_0) - e^{-(p_0)}}{\cos(p_0) + e^{-(p_0)}} \right] \\ & \Rightarrow p_1 = 6 - \left[\frac{\sin(6) - e^{-(6)}}{\cos(6) + e^{-(6)}} \right] = 6.2928318 \end{aligned}$$

$$\begin{aligned} \text{For } n = 2 & \Rightarrow p_2 = p_1 - \left[\frac{\sin(p_1) - e^{-(p_1)}}{\cos(p_1) + e^{-(p_1)}} \right] \\ & \Rightarrow p_2 = p_1 - \left[\frac{\sin(p_1) - e^{-(p_1)}}{\cos(p_1) + e^{-(p_1)}} \right] = 6.285049004 \end{aligned}$$

Continuing in this process, we can find a table-

Newton- Raphson Method			
Number of Iteration (n)	Approximation p_n	$E_R = \frac{ p_{\text{new}} - p_{\text{old}} }{ p_{\text{new}} }$	$E_P = E_R \cdot 100$
1	$p_1 = 6.2928318$	$E_R = \frac{ p_1 - p_0 }{ p_1 } =$	
2	$p_2 = 6.285049004$	$E_R = \frac{ p_2 - p_1 }{ p_2 } =$	
3	$p_3 = 6.285049273$	$E_R = \frac{ p_3 - p_2 }{ p_3 } =$	
4	$p_4 = 6.285049273$	$E_R = \frac{ p_4 - p_3 }{ p_4 } =$	

We observe that p_2 and p_3 are same up to Six decimal places. Thus, $p = 6.285049$ is correct root of the equation $f(x) = 0$ up to Six places of decimals.

Example 2.5.4: Apply **Newton-Raphson Method** to find the solution accurate to within 10^{-5} to the given problem: $\cos x - xe^x = 0$, for $0 \leq x \leq 1$

Answer: We have $f(x) = \cos x - xe^x$

Since, $f(0) = \cos(0) - (0)e^0 = 1 > 0$ (Positive)

and $f(1) = \cos(1) - (1)e^1 = -2.1779 < 0$ (Negative)

Therefore, by intermediate theorem, the function $f(x)$ has at least one real root between 0 and 1.

Now, $f(x) = \cos(x) - (x)e^{(x)} \Rightarrow f(p_{n-1}) = \cos(p_{n-1}) - (p_{n-1})e^{(p_{n-1})} \dots (1)$

$f'(x) = -\sin(x) - e^{(x)} - (x)e^{(x)} \Rightarrow f'(p_{n-1}) = -\sin(p_{n-1}) - e^{(p_{n-1})} - (p_{n-1})e^{(p_{n-1})} \dots (2)$

Applying Newton – Raphson Method-

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad \text{for all } n \geq 1$$

From Equations ...(1) and ...(2), we can write -

$$\Rightarrow p_n = p_{n-1} - \left[\frac{\cos(p_{n-1}) - (p_{n-1})e^{(p_{n-1})}}{-\sin(p_{n-1}) - e^{(p_{n-1})} - (p_{n-1})e^{(p_{n-1})}} \right] \quad \text{for all } n \geq 1$$

Choose, $p_0 = 0$, (Because $f(0) = +1$ is near to Zero in comparison of $f(1) = -2.1779$).

For $n = 1 \Rightarrow p_1 = p_0 + \left[\frac{\cos(p_0) - (p_0)e^{(p_0)}}{\sin(p_0) + e^{(p_0)} + (p_0)e^{(p_0)}} \right] = 1$

For $n = 2 \Rightarrow p_2 = p_1 + \left[\frac{\cos(p_1) - (p_1)e^{(p_1)}}{\sin(p_1) + e^{(p_1)} + (p_1)e^{(p_1)}} \right] = 0.6530794035$

Continuing in this process, we can find a table-

Newton- Raphson Method		
Number of Iteration (n)	Approximation $p_n = p_{n-1} - \left[\frac{\cos(p_{n-1}) - (p_{n-1})e^{(p_{n-1})}}{-\sin(p_{n-1}) - e^{(p_{n-1})} - (p_{n-1})e^{(p_{n-1})}} \right]$	$E_p = \frac{ p_{new} - p_{old} }{ p_{new} } \cdot 100$
1	$p_1 = 1$	$E_p = \frac{ p_1 - p_0 }{ p_1 } \cdot 100 =$
2	$p_2 = 0.6530794035$	$E_p = \frac{ p_2 - p_1 }{ p_2 } \cdot 100 =$
3	$p_3 = 0.5313433676$	$E_p = \frac{ p_3 - p_2 }{ p_3 } \cdot 100 =$
4	$p_4 = 0.5179099131$	$E_p = \frac{ p_4 - p_3 }{ p_4 } \cdot 100 =$
5	$p_5 = 0.5177573832$	$E_p = \frac{ p_5 - p_4 }{ p_5 } \cdot 100 =$
6	$p_6 = 0.5177573637$ STOP	$E_p = \frac{ p_6 - p_5 }{ p_6 } \cdot 100 =$

We observe that p_5 and p_6 are same up to **Seven** decimal places. Thus, $p = 0.5177573$ is correct root of the equation $f(x) = 0$ up to **Seven** places of decimals.

Example 2.5.5: Apply Newton- Raphson Method to approximate a root of the following nonlinear equation: (Do Three Iterations): $x^3 - 6x^2 + 14x - 6 = 0$ with $p_0 = 0.5$.

Solution: We have $f(x) = x^3 - 6x^2 + 14x - 6$

Since, $f(0) = (0)^3 - 6(0)^2 + 14(0) - 6 = -6 < 0$ (Negative)

and $f(1) = (1)^3 - 6(1)^2 + 14(1) - 6 = +3 > 0$ (Positive)

Therefore, by intermediate theorem, the function $f(x)$ has at least one real root between 0 and 1.

$$f(x) = x^3 - 6x^2 + 14x - 6 \Rightarrow f(p_{n-1}) = (p_{n-1})^3 - 6(p_{n-1})^2 + 14(p_{n-1}) - 6 \dots (1)$$

$$f'(x) = 3x^2 - 12x + 14 \Rightarrow f'(p_{n-1}) = 3(p_{n-1})^2 - 12(p_{n-1}) + 14 \dots (2)$$

Applying Newton-Raphson Method-

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad \text{for all } n \geq 1$$

From Equations (1) and (2), we can write as follows-

$$\Rightarrow p_n = p_{n-1} - \left[\frac{(p_{n-1})^3 - 6(p_{n-1})^2 + 14(p_{n-1}) - 6}{3(p_{n-1})^2 - 12(p_{n-1}) + 14} \right] \quad \text{for all } n \geq 1$$

$$\text{For, } n = 1 \quad p_1 = p_0 - \left[\frac{(p_0)^3 - 6(p_0)^2 + 14(p_0) - 6}{3(p_0)^2 - 12(p_0) + 14} \right] \quad p_0 = 0.5$$

$$\Rightarrow p_1 = 0.5 - \left[\frac{(0.5)^3 - 6(0.5)^2 + 14(0.5) - 6}{3(0.5)^2 - 12(0.5) + 14} \right] = \frac{19}{35} = 0.542857143$$

Newton- Raphson Method		
No of Iteration n	Approximation: $p_n = p_{n-1} - \left[\frac{(p_{n-1})^3 - 6(p_{n-1})^2 + 14(p_{n-1}) - 6}{3(p_{n-1})^2 - 12(p_{n-1}) + 14} \right]$	Percentage Error: $E_P = \frac{ p_{new} - p_{old} }{ p_{new} } \cdot 100$
1	$p_1 = 0.542857143$	
2	$p_2 = 0.543835254$	
3	$p_3 = 0.543835754$	

2.5 Secant Method:

We have seen that **Newton- Raphson method** is an extremely powerful technique to solve a nonlinear equation, but it has a major weakness that it requires the evaluation of derivative of the function at each approximation and this is not always possible, particularly in the case of functions arising in the practical problems.

To circumvent the problem of the derivative evaluation in **Newton- Raphson method**, we introduce a new technique known as **Secant Method**.

In the **Secant Method**, the derivative of $f(x)$ at p_i is approximated by:

$$f'(p_i) = \frac{f(p_i) - f(p_{i-1})}{p_i - p_{i-1}} \quad \dots (1)$$

Substitute $f'(p_i)$ from equation $\dots (1)$ in the **Newton-Raphson Formula**, we get-

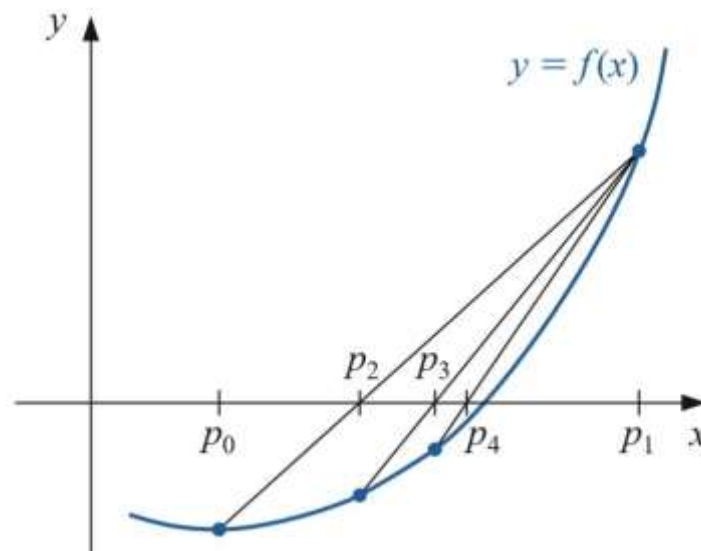
$$p_{i+1} = p_i - \frac{f(p_i)}{f'(p_i)} \quad \dots (2)$$

$$\Rightarrow p_{i+1} = p_i - \frac{f(p_i)}{\left[\frac{f(p_i) - f(p_{i-1})}{p_i - p_{i-1}} \right]}$$

$$\Rightarrow p_{i+1} = p_i - \left[\frac{(p_i - p_{i-1})}{f(p_i) - f(p_{i-1})} \right] f(p_i), \quad [i = 1, 2, 3, 4 \dots] \quad \dots (3)$$

$$\text{Or} \quad p_{i+1} = \frac{p_{i-1}f(p_i) - p_i f(p_{i-1})}{f(p_i) - f(p_{i-1})}, \quad [i = 1, 2, 3, 4 \dots] \quad \dots (4)$$

Starting with the two initial approximations p_0 and p_1 , the approximation p_2 is the **x-intercept** of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$. The approximation p_3 is the **x-intercept** of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$, and so on. Note that only one function evaluation is needed per step for the **Secant Method** after p_2 has been determined.



In contrast, each step of **Newton-Raphson** method requires an evaluation of both the function and its derivative

Example 2.6.1: Use the **Secant Method** to find a solution to $x = \cos x$ with $p_0 = 0.5$ and $p_1 = \frac{\pi}{4}$ with an accuracy of 10^{-7} . Find Relative Approximate Error Also.

Solution: Writing: $f(x) = \cos x - x$

$$f(0.5) = \cos(0.5) - 0.5 = 0.3775825 > 0 \quad (\text{Positive})$$

$$\text{and} \quad f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - \frac{\pi}{4} = -0.0782913 < 0 \quad (\text{Negative})$$

Therefore, the root lies between 0.5 and $\frac{\pi}{4} = 0.7853981635$

Now, Using **Secant Method**, starting with $p_0 = 0.5$ and $p_1 = \frac{\pi}{4}$

$$p_{n+1} = p_n - \left[\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})} \right] f(p_n) \quad \text{for all } n \geq 1$$

$$\text{Taking,} \quad f(x) = \cos x - x$$

$$\Rightarrow f(p_n) = \cos p_n - p_n \quad \text{and} \quad f(p_{n-1}) = \cos p_{n-1} - p_{n-1}$$

$$\text{Thus,} \quad p_{n+1} = p_n - \left[\frac{p_n - p_{n-1}}{[\cos p_n - p_n] - [\cos p_{n-1} - p_{n-1}]} \right] [\cos p_n - p_n] \quad \text{for all } n \geq 1$$

$$\Rightarrow p_2 = p_1 - \left[\frac{p_1 - p_0}{[\cos p_1 - p_1] - [\cos p_0 - p_0]} \right] [\cos p_1 - p_1]$$

$$p_2 = 0.7853981635 - \left[\frac{(0.7853981635 - 0.5)}{[\cos 0.7853981635 - 0.7853981635] - [\cos 0.5 - 0.5]} \right] [\cos 0.7853981635 - 0.7853981635]$$

$$\Rightarrow p_2 = 0.7363841388$$

$$\text{Next,} \quad p_3 = p_2 - \left[\frac{p_2 - p_1}{[\cos p_2 - p_2] - [\cos p_1 - p_1]} \right] [\cos p_2 - p_2]$$

Similarly, we can find p_3, p_4, p_5 etc. We list all in the following Table –

Number of Iteration (n)	Secant Method p_n	$E_R = \frac{ p_{\text{new}} - p_{\text{old}} }{ p_{\text{new}} }$	$E_P = E_R \cdot 100$
0	$p_0 = 0.5$		
1	$p_1 = 0.7853981635$		
2	$p_2 = 0.7363841388$	$E_R = \frac{ p_2 - p_1 }{ p_2 } =$	
3	$p_3 = 0.7390581392$	$E_R = \frac{ p_3 - p_2 }{ p_3 } =$	
4	$p_4 = 0.7390851493$	$E_R = \frac{ p_4 - p_3 }{ p_4 } =$	$3.6 \times 10^{-3} \%$
5	$p_5 = 0.7390851332$	$E_R = \frac{ p_5 - p_4 }{ p_5 } =$	$2.2 \times 10^{-6} \%$
6	STOP		

We observe that p_4 and p_5 are same up to **SEVEN** decimal places. Thus, $p = 0.7390851$ is correct root of the function $f(x)$ up to seven decimal places.

Example 2.6.2: Use the **Secant Method** to find a solution of $2x - \log_{10}(x) = 7$ on the interval $[3, 4]$ with an accuracy of 10^{-5} .

Solution: We have $f(x) = 2x - \log_{10}(x) - 7$ **Interval:** $[3, 4]$

Since, $f(3) = 2(3) - \log_{10}(3) - 7 = -1.47712125 < 0$ (**Negative**)

and $f(4) = 2(4) - \log_{10}(4) - 7 = +0.39794000 > 0$ (**Positive**)

$\Rightarrow f(x)$ must have one root between: 3 and 4.

Choose, $p_0 = 3$ and $p_1 = 4$

$\Rightarrow f(p_0) = -1.47712125$ & $f(p_1) = 0.39794000$

Applying Secant Method –

$$p_{n+1} = \frac{p_{n-1}f(p_n) - p_nf(p_{n-1})}{f(p_n) - f(p_{n-1})} \quad \text{for all } n \geq 1 \quad \dots\dots\dots (1)$$

Iteration-1: Taking $n = 1$ in equation-(1), we get-

$$p_2 = \frac{p_0f(p_1) - p_1f(p_0)}{f(p_1) - f(p_0)} \quad \dots\dots\dots (2)$$

$$\Rightarrow p_2 = 3.787772266 \quad \Rightarrow f(p_2) = -0.002839328$$

Iteration-2: Taking $n = 2$ in equation-(1), we get-

$$p_3 = \frac{p_1f(p_2) - p_2f(p_1)}{f(p_2) - f(p_1)} \quad \dots\dots\dots (3)$$

$$\Rightarrow p_3 = 3.789275797 \quad \Rightarrow f(p_3) = -0.000004622$$

Iteration-3: Taking $n = 3$ in equation-(1), we get-

$$p_4 = \frac{p_2f(p_3) - p_3f(p_2)}{f(p_3) - f(p_2)} \quad \dots\dots\dots (4)$$

$$\Rightarrow p_4 = 3.789278249 \quad \text{STOP}$$

We observe that: p_3 and p_4 are same up to **five** decimal places. Thus, $p = 3.78927$ is correct root of the given equation $f(x) = 0$ up to **five** places of decimals.

2.6 Numerical Solution of a System of Nonlinear Equations:

Consider a system of two nonlinear algebraic equations -

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases} \quad \dots (1)$$

(a) Newton's Iterative Technique OR Newton-Raphson Extension Method:

Let (x_0, y_0) be an initial approximation to the root of Equation (1). If $(x_0 + h, y_0 + k)$ is the exact root of the system, then we must have-

$$\begin{cases} f(x_0 + h, y_0 + k) = 0 \\ g(x_0 + h, y_0 + k) = 0 \end{cases} \quad \dots (2)$$

Assume that f and g are sufficiently differentiable functions, we can expand both the functions by Taylor's series to obtain-

$$\begin{cases} f_0 + h \frac{\partial f}{\partial x_0} + k \frac{\partial f}{\partial y_0} + \dots = 0 \\ g_0 + h \frac{\partial g}{\partial x_0} + k \frac{\partial g}{\partial y_0} + \dots = 0 \end{cases} \quad \dots (3)$$

Where,

$$f_0 = f(x_0, y_0)$$

$$\frac{\partial f}{\partial x_0} = \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)}$$

$$\frac{\partial f}{\partial y_0} = \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)}$$

$$g_0 = g(x_0, y_0)$$

$$\frac{\partial g}{\partial x_0} = \left. \frac{\partial g}{\partial x} \right|_{(x_0, y_0)}$$

$$\frac{\partial g}{\partial y_0} = \left. \frac{\partial g}{\partial y} \right|_{(x_0, y_0)}$$

Neglecting the second and higher order derivatives, we obtain the following-

$$\begin{cases} h \frac{\partial f}{\partial x_0} + k \frac{\partial f}{\partial y_0} = -f_0 \\ h \frac{\partial g}{\partial x_0} + k \frac{\partial g}{\partial y_0} = -g_0 \end{cases} \quad \dots (3)$$

Equation (3) possesses a unique solution if

$$D = \begin{vmatrix} \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial y_0} \\ \frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial y_0} \end{vmatrix} \neq 0 \quad \dots (4)$$

By **Cramer's Rule**, the solution of equation (3) is given by-

$$h = \frac{1}{D} \begin{vmatrix} -f_0 & \frac{\partial f}{\partial y_0} \\ -g_0 & \frac{\partial g}{\partial y_0} \end{vmatrix} \quad \text{and} \quad k = \frac{1}{D} \begin{vmatrix} \frac{\partial f}{\partial x_0} & -f_0 \\ \frac{\partial g}{\partial x_0} & -g_0 \end{vmatrix} \quad \dots (5)$$

Hence the new approximations are therefore,

$$x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + k \quad \dots (6)$$

The process is to be repeated till we obtain the roots to the desired accuracy.

Note: (1) Choose initial approximation if not given in the question: $(x_0, y_0) = (0, 0)$ or $(x_0, y_0) = (0.5, 0.5)$ or $(x_0, y_0) = (1, 1)$

(2) The convergence of the **Newton's Iteration** method for solving a system of nonlinear equations is of second order, same as in case of single nonlinear equation.

Example 2.7.1: Use **Newton's Iterative Techniques** to find approximate solutions up to 2nd iteration to the following nonlinear **system** of equations: [Choose: $(x_0, y_0) = (0.5, 0.5)$]

$$\begin{cases} 3x^2y - 10x + 7 = 0 \\ y^2 - 5y + 4 = 0 \end{cases}$$

Solution: We have $f(x, y) = 3x^2y - 10x + 7$

$$\Rightarrow \frac{\partial f}{\partial x} = 6xy - 10 \quad \& \quad \frac{\partial f}{\partial y} = 3x^2$$

$$g(x, y) = y^2 - 5y + 4$$

$$\Rightarrow \frac{\partial g}{\partial x} = 0 \quad \& \quad \frac{\partial g}{\partial y} = 2y - 5$$

First Approximation: Taking, $(x_0, y_0) = (0.5, 0.5)$, we obtain-

$$f_0 = f(x_0, y_0) = 2.375, \quad g_0 = g(x_0, y_0) = 1.75$$

$$\frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = -8.5, \quad \frac{\partial f}{\partial y_0} = \frac{\partial f}{\partial y} \Big|_{(x_0, y_0)} = 0.75$$

$$\frac{\partial g}{\partial x_0} = \frac{\partial g}{\partial x} \Big|_{(x_0, y_0)} = 0, \quad \frac{\partial g}{\partial y_0} = \frac{\partial g}{\partial y} \Big|_{(x_0, y_0)} = -4$$

$$\left. \begin{aligned} h \frac{\partial f}{\partial x_0} + k \frac{\partial f}{\partial y_0} &= -f_0 \\ h \frac{\partial g}{\partial x_0} + k \frac{\partial g}{\partial y_0} &= -g_0 \end{aligned} \right\} \Rightarrow \begin{aligned} -8.5h + 0.75k &= -2.375 \\ 0h - 4k &= -1.75 \end{aligned} \quad \dots (3)$$

Cramer's Rule:

Hence,
$$D = \begin{vmatrix} \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial y_0} \\ \frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial y_0} \end{vmatrix} = \begin{vmatrix} -8.5 & 0.75 \\ 0 & -4 \end{vmatrix} = 34 \neq 0$$

By **Cramer's Rule**, the solution of equation (3) is given by-

$$h = \frac{1}{D} \begin{vmatrix} -f_0 & \frac{\partial f}{\partial y_0} \\ -g_0 & \frac{\partial g}{\partial y_0} \end{vmatrix} = \frac{1}{34} \begin{vmatrix} -2.375 & 0.75 \\ -1.75 & -4 \end{vmatrix} = 0.3180$$

$$\text{and } k = \frac{1}{D} \begin{vmatrix} \frac{\partial f}{\partial x_0} & -f_0 \\ \frac{\partial g}{\partial x_0} & -g_0 \end{vmatrix} = \frac{1}{34} \begin{vmatrix} -8.5 & -2.375 \\ 0 & -1.75 \end{vmatrix} = 0.4375$$

Hence the new approximations are therefore,

$$x_1 = x_0 + h = 0.5 + 0.3180 = 0.8180$$

$$y_1 = y_0 + k = 0.5 + 0.4375 = 0.9375$$

Second Approximation: Taking, $(x_1, y_1) = (0.8180, 0.9375)$, we obtain-

$$f_1 = f(x_1, y_1) = 0.7019$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x} \Big|_{(x_1, y_1)} = -5.3988$$

$$\frac{\partial g}{\partial x_1} = \frac{\partial g}{\partial x} \Big|_{(x_1, y_1)} = 0$$

$$g_1 = g(x_1, y_1) = 0.1914$$

$$\frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial y} \Big|_{(x_1, y_1)} = 2.0074$$

$$\frac{\partial g}{\partial y_1} = \frac{\partial g}{\partial y} \Big|_{(x_1, y_1)} = -3.125$$

$$\left. \begin{aligned} h \frac{\partial f}{\partial x_1} + k \frac{\partial f}{\partial y_1} &= -f_1 \\ h \frac{\partial g}{\partial x_1} + k \frac{\partial g}{\partial y_1} &= -g_1 \end{aligned} \right\} \Rightarrow \begin{aligned} -5.3988h + 2.0074k &= -0.7019 \\ 0h - 3.125k &= -0.1914 \end{aligned}$$

Hence,
$$D = \begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial y_1} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial y_1} \end{vmatrix} = \begin{vmatrix} -5.3988 & 2.0074 \\ 0 & -3.125 \end{vmatrix} = 16.8712 \neq 0$$

By Crammer's Rule, the solution of equation (3) is given by-

$$h = \frac{1}{D} \begin{vmatrix} -f_1 & \frac{\partial f}{\partial y_1} \\ -g_1 & \frac{\partial g}{\partial y_1} \end{vmatrix} = \frac{1}{16.8712} \begin{vmatrix} -0.7019 & 2.0074 \\ -0.1914 & -3.125 \end{vmatrix} = 0.1528$$

and
$$k = \frac{1}{D} \begin{vmatrix} \frac{\partial f}{\partial x_1} & -f_1 \\ \frac{\partial g}{\partial x_1} & -g_1 \end{vmatrix} = \frac{1}{16.8712} \begin{vmatrix} -5.3988 & -0.7019 \\ 0 & -0.1914 \end{vmatrix} = 0.0612$$

It follows that,

$$x_2 = x_1 + h = 0.8180 + 0.1528 = 0.9708$$

$$y_2 = y_1 + k = 0.9375 + 0.0612 = 0.9987$$

Example 2.7.2: Use **Newton's Iterative Techniques** to find approximate solutions up to 2nd iteration to the given nonlinear **system** of equations: [Starting with $(x_0, y_0) = (0.5, 1.5)$]

$$\begin{cases} \sin x - y = -0.9793 \\ \cos y - x = -0.6703 \end{cases}$$

Solution: We have $f(x, y) = \sin x - y + 0.9793$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \cos x \quad \& \quad \frac{\partial f}{\partial y} = -1$$

$$g(x, y) = \cos y - x + 0.6703$$

$$\Rightarrow \quad \frac{\partial g}{\partial x} = -1 \quad \& \quad \frac{\partial g}{\partial y} = -\sin y$$

First Approximation: Taking, $(x_0, y_0) = (0.5, 1.5)$, we obtain-

$$f_0 = f(x_0, y_0) = -0.0413, \quad g_0 = g(x_0, y_0) = 0.2410$$

$$\frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = 0.8776, \quad \frac{\partial f}{\partial y_0} = \frac{\partial f}{\partial y} \Big|_{(x_0, y_0)} = -1$$

$$\frac{\partial g}{\partial x_0} = \frac{\partial g}{\partial x} \Big|_{(x_0, y_0)} = -1, \quad \frac{\partial g}{\partial y_0} = \frac{\partial g}{\partial y} \Big|_{(x_0, y_0)} = -0.9975$$

$$\left. \begin{aligned} h \frac{\partial f}{\partial x_0} + k \frac{\partial f}{\partial y_0} &= -f_0 \\ h \frac{\partial g}{\partial x_0} + k \frac{\partial g}{\partial y_0} &= -g_0 \end{aligned} \right\} \Rightarrow \quad \begin{cases} 0.8776h - k = 0.0413 \\ -h - 0.9975k = -0.2410 \end{cases} \quad \dots (3)$$

Cramer's Rule:

Hence,
$$D = \begin{vmatrix} \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial y_0} \\ \frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial y_0} \end{vmatrix} = \begin{vmatrix} 0.8776 & -1 \\ -1 & -0.9975 \end{vmatrix} = -1.8754 \neq 0$$

By **Cramer's Rule**, the solution of equation (3) is given by-

$$h = \frac{1}{D} \begin{vmatrix} -f_0 & \frac{\partial f}{\partial y_0} \\ -g_0 & \frac{\partial g}{\partial y_0} \end{vmatrix} = \frac{1}{-1.8754} \begin{vmatrix} -0.0413 & -1 \\ 0.2410 & -0.9975 \end{vmatrix} = -0.1505$$

$$\text{and } k = \frac{1}{D} \begin{vmatrix} \frac{\partial f}{\partial x_0} & -f_0 \\ \frac{\partial g}{\partial x_0} & -g_0 \end{vmatrix} = \frac{1}{-1.8754} \begin{vmatrix} 0.8776 & -0.0413 \\ -1 & 0.2410 \end{vmatrix} = -0.0908$$

Hence the new approximations are therefore,

$$x_1 = x_0 + h = 0.5 + 0.1505 = 0.6505$$

$$y_1 = y_0 + k = 1.5 + 0.0908 = 1.5908$$

Second Approximation: Taking, $(x_1, y_1) = (0.6505, 1.5908)$, we obtain-

$$f_1 = f(x_1, y_1) = -0.00592$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x} \Big|_{(x_1, y_1)} = 0.7958$$

$$\frac{\partial g}{\partial x_1} = \frac{\partial g}{\partial x} \Big|_{(x_1, y_1)} = -1$$

$$g_1 = g(x_1, y_1) = -0.00020$$

$$\frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial y} \Big|_{(x_1, y_1)} = -1$$

$$\frac{\partial g}{\partial y_1} = \frac{\partial g}{\partial y} \Big|_{(x_1, y_1)} = -0.9998$$

$$\left. \begin{aligned} h \frac{\partial f}{\partial x_1} + k \frac{\partial f}{\partial y_1} &= -f_1 \\ h \frac{\partial g}{\partial x_1} + k \frac{\partial g}{\partial y_1} &= -g_1 \end{aligned} \right\} \Rightarrow \begin{aligned} 0.7958h - 1k &= 0.00592 \\ -h - 0.9998k &= 0.00020 \end{aligned}$$

Hence,
$$D = \begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial y_1} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial y_1} \end{vmatrix} = \begin{vmatrix} 0.7958 & -1 \\ -1 & -0.9998 \end{vmatrix} = -1.7956 \neq 0$$

By Crammer's Rule, the solution of equation (3) is given by-

$$h = \frac{1}{D} \begin{vmatrix} -f_1 & \frac{\partial f}{\partial y_1} \\ -g_1 & \frac{\partial g}{\partial y_1} \end{vmatrix} = \frac{1}{-1.7956} \begin{vmatrix} 0.00592 & -1 \\ 0.00020 & -0.9998 \end{vmatrix} = 0.003181$$

and
$$k = \frac{1}{D} \begin{vmatrix} \frac{\partial f}{\partial x_1} & -f_1 \\ \frac{\partial g}{\partial x_1} & -g_1 \end{vmatrix} = \frac{1}{-1.7956} \begin{vmatrix} 0.7958 & 0.00592 \\ -1 & 0.00020 \end{vmatrix} = 0.003384$$

It follows that,

$$x_2 = x_1 + h = 0.6505 + 0.0032 = 0.6537$$

$$y_2 = y_1 + k = 1.5908 - 0.0034 = 1.5874$$

Example 2.7.3: Solve the following system of **nonlinear algebraic** equations with initial approximation, $(x_0, y_0) = (1, 1)$ (**Do One iteration**):

$$\begin{cases} xy^3 - 3y = -2 \\ -x^2y + 6x = -3 \end{cases}$$

Solution: We have

$$f(x, y) = xy^3 - 3y + 2 \Rightarrow \frac{\partial f}{\partial x} = y^3 \quad \& \quad \frac{\partial f}{\partial y} = 3xy^2 - 3 \quad \dots (1)$$

$$g(x, y) = -x^2y + 6x + 3 \Rightarrow \frac{\partial g}{\partial x} = -2xy + 6 \quad \& \quad \frac{\partial g}{\partial y} = -x^2 \quad \dots (2)$$

First Approximation: Taking, $(x_0, y_0) = (1, 1)$, From Equations ... (1) & (2):

$$f_0 = f(x_0, y_0) = 0,$$

$$g_0 = g(x_0, y_0) = 8$$

$$\frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = 1,$$

$$\frac{\partial f}{\partial y_0} = \frac{\partial f}{\partial y} \Big|_{(x_0, y_0)} = 0$$

$$\frac{\partial g}{\partial x_0} = \frac{\partial g}{\partial x} \Big|_{(x_0, y_0)} = 4,$$

$$\frac{\partial g}{\partial y_0} = \frac{\partial g}{\partial y} \Big|_{(x_0, y_0)} = -1$$

$$\begin{cases} h \frac{\partial f}{\partial x_0} + k \frac{\partial f}{\partial y_0} = -f_0 \\ h \frac{\partial g}{\partial x_0} + k \frac{\partial g}{\partial y_0} = -g_0 \end{cases} \Rightarrow \begin{cases} 1h + 0k = 0 \\ 4h - 1k = -8 \end{cases} \quad \dots (3)$$

Using Calculator:

$$h = 0$$

$$k = 8$$

Using Crammer's Rule:

Hence,
$$D = \begin{vmatrix} \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial y_0} \\ \frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial y_0} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 4 & -1 \end{vmatrix} = -1 \neq 0$$

By Crammer's Rule, the solution of equation (3) is given by-

$$h = \frac{1}{D} \begin{vmatrix} -f_0 & \frac{\partial f}{\partial y_0} \\ -g_0 & \frac{\partial g}{\partial y_0} \end{vmatrix} = \frac{1}{-1} \begin{vmatrix} 0 & 0 \\ -8 & -1 \end{vmatrix} = 0$$

and
$$k = \frac{1}{D} \begin{vmatrix} \frac{\partial f}{\partial x_0} & -f_0 \\ \frac{\partial g}{\partial x_0} & -g_0 \end{vmatrix} = \frac{1}{-1} \begin{vmatrix} 1 & 0 \\ 4 & -8 \end{vmatrix} = 8$$

Hence the new approximations are therefore,

$$x_1 = x_0 + h = 1 + 0 = 1$$

$$y_1 = y_0 + k = 1 + 8 = 9$$

Exercise 2.1: Numerical Solution of Nonlinear Equations

1. Choose the correct answer (Questions 1.1 -1.5)

1.1 Which of the following is a **system of linear equations**:

(a) $x^3 - 2x^2 - 5 = 0$

(b)
$$\left. \begin{aligned} \sin x - y &= -0.9793 \\ \cos y - x &= -0.6703 \end{aligned} \right\}$$

(c)
$$\left. \begin{aligned} 3x - 10y &= 7 \\ 7x + 5y &= 4 \end{aligned} \right\}$$

(d) None of these

1.2 Which of the following formula is regarded as the **Relative Approximate Error (E_R)**:

(a) $|P_{New} - P_{Old}|$

(b) $\frac{|P_{New} - P_{Old}|}{|P_{New}|}$

(c) $\frac{|P_{New} - P_{Old}|}{|P_{New}|} \cdot 100$

(d) None of these

1.3 A real root of the equation: $2x - \log_{10}(x) = 7$ accurate to within 10^{-4} is

(a) 1.78927

(b) 2.78927

(c) 3.78927

(d) None of these

1.4 Distinguish between **intervals** $[-2, -1]$ and $[-1, 1]$ for the given nonlinear equation $x^2 + 4\sin x = 0$ to have at least one real root:

- (a) $[-2, -1]$
- (b) $[-1, 1]$
- (c) Both (a) and (b)
- (d) None of these

1.5 The **Newton-Raphson Method** **fails** to find a root of a given **nonlinear equation** $f(x) = 0$, if:

- (a) $f'(P_n) > 1$
- (b) $f'(P_n) = 0$
- (c) $f'(P_n) < 0$
- (d) $f''(P_n) = 0$

2. Find the interval $[a, b]$ in which, the given equation $f(x) = 0$ has at least one real root.

- (i) $x^3 + 4x^2 - 10 = 0$
- (ii) $\sin x - e^{-x} = 0$
- (iii) $x^3 - 2x^2 - 5 = 0$
- (iv) $2x - \log_{10}(x) = 12$

3. Find an approximation to $\sqrt{3}$ correct to within 10^{-4} using the **Bisection Method**. [Hint: Consider $f(x) = x^2 - 3$]

4. Use **the Bisection Method** to find an approximation to $\sqrt{30}$ that is accurate to within 10^{-4} .

5. Use **the Bisection Method** to determine a solution of the equation $x^3 - 7x^2 + 14x - 6 = 0$ on the interval $[0, 1]$ with an accuracy of 10^{-2} .

6. Use **the Bisection Method** to find a solution of the equation $3x - e^x = 0$ on the interval $[1, 2]$ with an accuracy of 10^{-4} .

7. Use **the Newton- Raphson Method** to find a solution accurate to within 10^{-4} for the following equation:

$$x^3 - 2x^2 - 5 = 0, \quad [1, 4]$$

8. Use **the Newton- Raphson Method** to find a solution accurate to within 10^{-4} for the following equation:

$$\sin x - e^{-x} = 0, \quad \text{for } 0 \leq x \leq 1$$

9. Use **the Newton- Raphson Method** to find a solution accurate to within 10^{-4} for the following equation:

$$x - \cos x = 0, \quad \left[0, \frac{\pi}{2}\right]$$

10. Use **the Secant Method** to find a solution accurate to within 10^{-4} for the following equation:

$$x^3 - 2x^2 - 5 = 0, \quad [1, 4]$$

11. Use **the Secant Method** to find a solution accurate to within 10^{-4} for the following equation:

$$\sin x - e^{-x} = 0, \quad \text{for } 0 \leq x \leq 1$$

12. Use **the Secant Method** to find a solution accurate to within 10^{-4} for the following equation:

$$x - \cos x = 0, \quad \left[0, \frac{\pi}{2}\right]$$

13. Use **Newton's Iterative Techniques** to find approximate solutions to the following nonlinear **system** of equations:

$$(a) \quad \left. \begin{array}{l} x^2 + y^2 = 1 \\ y = x^3 \end{array} \right\} \quad \text{Taking, } (x_0, y_0) = (1, 1)$$

$$(b) \quad \left. \begin{array}{l} x^2 - y^2 = 4 \\ x^2 + y^2 = 16 \end{array} \right\} \quad \text{Taking, } (x_0, y_0) = (3.0, 2.5)$$

$$(c) \quad \left. \begin{array}{l} x^2 + y = 11 \\ x + y^2 = 7 \end{array} \right\} \quad \text{Taking, } (x_0, y_0) = (0.75, 0.75)$$

Numerical Analysis-1

(For Science College Students)

MATH-316

Chapter-3: Interpolation

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Chapter-3: Interpolation

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3.1 Interpolation:

Interpolation is the technique of obtaining the value of a function for any intermediate value of the intermediate variable.

3.2 Finite Difference Operators:

(a) Forward Difference Operators:

Let $y = f(x)$, $x_0 \leq x \leq x_n$ be a function then we can write:

x	x_0	x_1	x_2	x_3	$\dots$	x_n
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\dots$	$f(x_n)$
y	y_0	y_1	y_2	y_3		y_n

First Forward Difference Operator: Δ

The first forward difference Δ of the function $y = f(x)$ is defined as:

$$\Delta f(x) = f(x + h) - f(x)$$

$$\begin{aligned} \Rightarrow \Delta f(x_i) &= f(x_{i+1}) - f(x_i), & \text{Or} & \Delta y_i = y_{i+1} - y_i, & i = 0, 1, 2, 3, \dots (n-1) \\ \Rightarrow \Delta f(x_0) &= f(x_1) - f(x_0), & \text{Or} & \Delta y_0 = y_1 - y_0 \\ \Rightarrow \Delta f(x_1) &= f(x_2) - f(x_1), & \text{Or} & \Delta y_1 = y_2 - y_1 \\ \Rightarrow \Delta f(x_2) &= f(x_3) - f(x_2), & \text{Or} & \Delta y_2 = y_3 - y_2 \\ & \dots & & \dots & \\ \Rightarrow \Delta f(x_{n-1}) &= f(x_n) - f(x_{n-1}), & \text{Or} & \Delta y_{n-1} = y_n - y_{n-1} \end{aligned}$$

Second Forward Difference Operator: Δ^2

The Second forward difference Δ^2 of the function $y = f(x)$ is defined as:

$$\begin{aligned} \Delta^2 f(x_i) &= \Delta[\Delta f(x_i)] \\ &= \Delta f(x_{i+1}) - \Delta f(x_i) \\ &= [f(x_{i+2}) - f(x_{i+1})] - [f(x_{i+1}) - f(x_i)] \\ &= f(x_{i+2}) - 2f(x_{i+1}) + f(x_i) \end{aligned}$$

Similarly, we can define third and higher order forward difference for the function $y = f(x)$ as

$$\begin{aligned} \Delta^3 f(x_i) &= \Delta^2[\Delta f(x_i)] \\ &= \Delta^2 f(x_{i+1}) - \Delta^2 f(x_i) \end{aligned}$$

$$\begin{aligned} \Delta^4 f(x_i) &= \Delta^3[\Delta f(x_i)] \\ &= \Delta^3 f(x_{i+1}) - \Delta^3 f(x_i) \quad \text{and so on} \end{aligned}$$

Forward Difference Table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
x_0	$f(x_0)$	$\Delta f(x_0)$	$\Delta^2 f(x_0)$	$\Delta^3 f(x_0)$	$\Delta^4 f(x_0)$	$\Delta^5 f(x_0)$
x_1	$f(x_1)$	$\Delta f(x_1)$	$\Delta^2 f(x_1)$	$\Delta^3 f(x_1)$	$\Delta^4 f(x_1)$	
x_2	$f(x_2)$	$\Delta f(x_2)$	$\Delta^2 f(x_2)$	$\Delta^3 f(x_2)$		
x_3	$f(x_3)$	$\Delta f(x_3)$	$\Delta^2 f(x_3)$			
x_4	$f(x_4)$	$\Delta f(x_4)$				
x_5	$f(x_5)$					

Example 3.2.1: Construct **Forward Difference Table** for the following data.

Years: x	1941	1951	1961	1971	1981	1991
Population(in million): $f(x)$	20	24	29	36	46	51

Find also the following values from **Forward Difference Table**:

- (a) $\Delta f(1961)$ (b) $\Delta^2 f(1971)$ (c) $\Delta^3 f(1961)$ (d) $\Delta^4 f(1951)$ (e) $\Delta^5 f(1941)$

Solution: Let us construct **forward difference table**:

Forward Difference Table							
	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
x_0	1941	20	4	1	1	0	-9
x_1	1951	24	5	2	1	-9	
x_2	1961	29	7	3	-8		
x_3	1971	36	10	-5			
x_4	1981	46	5				
x_5	1991	51					

From the Table, we can find-

- (a) $\Delta f(1961) = 7$
 (b) $\Delta^2 f(1971) = -5$
 (c) $\Delta^3 f(1961) = -8$
 (d) $\Delta^4 f(1951) = -9$
 (e) $\Delta^5 f(1941) = -9$

Example 3.2.2: If $f(1) = 4$, $f(2) = 8$, $f(3) = 12$, $f(4) = 18$ and $f(5) = 36$, then $\Delta^3 f(1)$ is -

- (a) 10 (b) 8 (c) 2 (d) None of these

Solution: Do yourself

(b) Backward Difference Operators:

Let $y = f(x)$, $x_0 \leq x \leq x_n$ be a function then we can write:

x	x_0	x_1	x_2	x_3	$\dots$	x_n
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\dots$	$f(x_n)$
y	y_0	y_1	y_2	y_3		y_n

First Backward Difference Operator: ∇

The first backward difference ∇ of the function $y = f(x)$ is defined as:

$$\nabla f(x) = f(x) - f(x - h)$$

$$\Rightarrow \boxed{\nabla f(x_i) = f(x_i) - f(x_{i-1})}, \quad \text{Or} \quad \boxed{\nabla y_i = y_i - y_{i-1}}, \quad i = n, (n-1), (n-2), \dots, 3, 2, 1$$

$$\Rightarrow \nabla f(x_n) = f(x_n) - f(x_{n-1}) \quad \text{Or} \quad \nabla y_n = y_n - y_{n-1}$$

$$\dots \dots \dots$$

$$\Rightarrow \nabla f(x_3) = f(x_3) - f(x_2) \quad \text{Or} \quad \nabla y_3 = y_3 - y_2$$

$$\Rightarrow \nabla f(x_2) = f(x_2) - f(x_1) \quad \text{Or} \quad \nabla y_2 = y_2 - y_1$$

$$\Rightarrow \nabla f(x_1) = f(x_1) - f(x_0) \quad \text{Or} \quad \nabla y_1 = y_1 - y_0$$

Second Backward Difference Operator: ∇^2

The Second backward difference ∇^2 of the function $y = f(x)$ is defined as:

$$\begin{aligned} \nabla^2 f(x_i) &= \nabla[\nabla f(x_i)] \\ &= \nabla f(x_i) - \nabla f(x_{i-1}) \\ &= [f(x_i) - f(x_{i-1})] - [f(x_{i-1}) - f(x_{i-2})] \\ &= f(x_i) - 2f(x_{i-1}) + f(x_{i-2}) \end{aligned}$$

Similarly, we can define third and higher order backward difference of the function $y = f(x)$ as

$$\begin{aligned} \nabla^3 f(x_i) &= \nabla^2[\nabla f(x_i)] \\ &= \nabla^2 f(x_i) - \nabla^2 f(x_{i-1}) \end{aligned}$$

$$\begin{aligned} \nabla^4 f(x_i) &= \nabla^3[\nabla f(x_i)] \\ &= \nabla^3 f(x_i) - \nabla^3 f(x_{i-1}) \end{aligned} \quad \text{and so on}$$

Backward Difference Table:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
x_0	$f(x_0)$					
x_1	$f(x_1)$	$\nabla f(x_1)$				
x_2	$f(x_2)$	$\nabla f(x_2)$	$\nabla^2 f(x_2)$			
x_3	$f(x_3)$	$\nabla f(x_3)$	$\nabla^2 f(x_3)$	$\nabla^3 f(x_3)$		
x_4	$f(x_4)$	$\nabla f(x_4)$	$\nabla^2 f(x_4)$	$\nabla^3 f(x_4)$	$\nabla^4 f(x_4)$	
x_5	$f(x_5)$	$\nabla f(x_5)$	$\nabla^2 f(x_5)$	$\nabla^3 f(x_5)$	$\nabla^4 f(x_5)$	$\nabla^5 f(x_5)$

Example 3.2.3: Construct **Backward Difference Table** for the following data.

Years: x	1941	1951	1961	1971	1981	1991
Population(in million): $f(x)$	20	24	29	36	46	51

Solution: Let us construct **backward difference table**:

Backward Difference Table							
	x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
x_0	1941	20					
x_1	1951	24	4				
x_2	1961	29	5	1			
x_3	1971	36	7	2	1		
x_4	1981	46	10	3	1	0	
x_5	1991	51	5	-5	-8	-9	-9

Find also the following values:

- (a) $\nabla f(1961)$ (b) $\nabla^2 f(1971)$ (c) $\nabla^3 f(1991)$ (d) $\nabla^4 f(1981)$ (e) $\nabla^5 f(1991)$

Answer: Do Yourself!

Example 3.2.4: If $f(1) = 4$, $f(2) = 8$, $f(3) = 12$, $f(4) = 18$ and $f(5) = 36$, then $\nabla^3 f(5)$ is -

- (a) 10 (b) 8 (c) 2 (d) None of these

Answer: We have

x	1	2	3	4	5
$f(x)$	4	8	12	18	36

Thus, $h = 1$ $n = 4$

Let us construct **backward difference table**:

Backward Difference Table						
	x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
x_0	1	4				
x_1	2	8	4			
x_2	3	12	4	0		
x_3	4	18	6	2	2	
x_4	5	36	18	12	10	0

From the table: $\nabla^3 f(5) = 10$

(c) Central Difference Operators:

Let $y = f(x)$, $x_0 \leq x \leq x_n$ be a function given by the following table:

x	x_0	x_1	x_2	x_3	$\dots$	x_n
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\dots$	$f(x_n)$
$y = f(x)$	$f(x_0)$	$f(x_0 + h)$	$f(x_0 + 2h)$	$f(x_0 + 3h)$	$\dots$	$f(x_0 + nh)$

The first order central difference operator δ is defined by the relations:

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$$\Rightarrow \delta f\left(x_0 + \frac{h}{2}\right) = f(x_0 + h) - f(x_0)$$

$$\Rightarrow \delta f\left(x_0 + \frac{3h}{2}\right) = f(x_0 + 2h) - f(x_0 + h)$$

$$\dots \dots \dots$$

$$\Rightarrow \delta f\left(x_0 + \frac{(2n-1)h}{2}\right) = f(x_0 + nh) - f(x_0 + (n-1)h)$$

Similarly, the higher order central difference operators are defined as:

$$\text{Since, } \delta f(x_0 + h) = f\left(x_0 + \frac{3h}{2}\right) - f\left(x_0 + \frac{h}{2}\right)$$

$$\Rightarrow \delta^2 f(x_0 + h) = \delta f\left(x_0 + \frac{3h}{2}\right) - \delta f\left(x_0 + \frac{h}{2}\right)$$

$$\Rightarrow \delta^2 f(x_0 + 2h) = \delta f\left(x_0 + \frac{5h}{2}\right) - \delta f\left(x_0 + \frac{3h}{2}\right) \quad \text{and so on....}$$

$$\Rightarrow \delta^3 f\left(x_0 + \frac{3h}{2}\right) = \delta^2 f(x_0 + 2h) - \delta^2 f(x_0 + h) \quad \text{and so on....}$$

$$\Rightarrow \delta^4 f(x_0 + 2h) = \delta^3 f\left(x_0 + \frac{5h}{2}\right) - \delta^3 f\left(x_0 + \frac{3h}{2}\right) \quad \text{and so on....}$$

$$\Rightarrow \delta^5 f\left(x_0 + \frac{5h}{2}\right) = \delta^4 f(x_0 + 3h) - \delta^4 f(x_0 + 2h)$$

Central Difference Table:

x	$f(x)$	$\delta f(x)$	$\delta^2 f(x)$	$\delta^3 f(x)$	$\delta^4 f(x)$	$\delta^5 f(x)$
x_0	$f(x_0)$					
		$\delta f\left(x_0 + \frac{h}{2}\right)$				
x_1	$f(x_1)$		$\delta^2 f(x_0 + h)$			
		$\delta f\left(x_0 + \frac{3h}{2}\right)$		$\delta^3 f\left(x_0 + \frac{3h}{2}\right)$		
x_2	$f(x_2)$		$\delta^2 f(x_0 + 2h)$		$\delta^4 f(x_0 + 2h)$	
		$\delta f\left(x_0 + \frac{5h}{2}\right)$		$\delta^3 f\left(x_0 + \frac{5h}{2}\right)$		$\delta^5 f\left(x_0 + \frac{5h}{2}\right)$
x_3	$f(x_3)$		$\delta^2 f(x_0 + 3h)$		$\delta^4 f(x_0 + 3h)$	
		$\delta f\left(x_0 + \frac{7h}{2}\right)$		$\delta^3 f\left(x_0 + \frac{7h}{2}\right)$		
x_4	$f(x_4)$		$\delta^2 f(x_0 + 4h)$			
		$\delta f\left(x_0 + \frac{9h}{2}\right)$				
x_5	$f(x_5)$					

Example 3.2.5: Construct **Central Difference Table** for the following data.

Years: x	1941	1951	1961	1971	1981	1991
Population(in million): $f(x)$	20	24	29	36	46	51

Find also the following values:

- (a) $\delta f(1966)$ (b) $\delta^2 f(1971)$ (c) $\delta^3 f(1976)$ (d) $\delta^4 f(1961)$ (e) $\delta^5 f(1966)$

Solution: Let us construct **central difference table**:

Central Difference Table							
	x	$f(x)$	$\delta f(x)$	$\delta^2 f(x)$	$\delta^3 f(x)$	$\delta^4 f(x)$	$\delta^5 f(x)$
x_0	1941	20					
$x_0 + \frac{1}{2}h$			4				
x_1	1951	24		1			
$x_0 + \frac{3}{2}h$			5		1		
x_2	1961	29		2		0	
$x_0 + \frac{5}{2}h$			7		1		-9
x_3	1971	36		3		-9	
$x_0 + \frac{7}{2}h$			10		-8		
x_4	1981	46		-5			
$x_0 + \frac{9}{2}h$			5				
x_5	1991	51					

From the Table, we can find-

- (a) $\delta f(1966) = 7$
 (b) $\delta^2 f(1971) =$
 (c) $\delta^3 f(1976) =$
 (d) $\delta^4 f(1961) =$
 (e) $\delta^5 f(1966) =$

(d) Shift Operator: E

Let $y = f(x)$ be a function given by:

x	x_0	$x_0 + h$	$x_0 + 2h$	$x_0 + 3h$	$\dots$	$x_0 + nh$
$y = f(x)$	$f(x_0)$	$f(x_0 + h)$	$f(x_0 + 2h)$	$f(x_0 + 3h)$	$\dots$	$f(x_0 + nh)$

The shift operator E is defined as:

$$Ef(x) = f(x + h)$$

and

$$E^{-1}f(x) = f(x - h)$$

Similarly, the higher order shift operators are defined as:

$$\Rightarrow E^2f(x) = f(x + 2h)$$

and

$$E^{-2}f(x) = f(x - 2h)$$

$$\Rightarrow E^3f(x) = f(x + 3h)$$

and

$$E^{-3}f(x) = f(x - 3h)$$

$$\dots$$

$$\dots$$

$$\Rightarrow E^n f(x) = f(x + nh)$$

$$E^{-n} f(x) = f(x - nh)$$

$$\Rightarrow E^{\frac{1}{2}}f(x) = f\left(x + \frac{1}{2}h\right)$$

and

$$E^{-\frac{1}{2}}f(x) = f\left(x - \frac{1}{2}h\right)$$

(e) Average or Mean Operators

The average or mean operator μ is defined by the following relation:

$$\mu f(x) = \frac{1}{2} \left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right]$$

3.3 Important Relations between Operators:

(a) Relationship between Δ and E:

We know that $\Delta f(x) = f(x + h) - f(x)$

$$\Rightarrow \Delta f(x) = Ef(x) - f(x)$$

$$\Rightarrow \Delta f(x) = (E - 1)f(x) \Rightarrow$$

$$\Delta \equiv E - 1$$

$\Rightarrow$

$$E \equiv 1 + \Delta$$

(b) Relationship between ∇ and E:

We know that $\nabla f(x) = f(x) - f(x - h)$

$$\Rightarrow \nabla f(x) = f(x) - E^{-1}f(x)$$

$$\Rightarrow \nabla f(x) = (1 - E^{-1})f(x) \Rightarrow$$

$$\nabla \equiv 1 - E^{-1}$$

$\Rightarrow$

$$E^{-1} \equiv 1 - \nabla$$

(c) Relationship between δ and E :

We know that $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$

$$\Rightarrow \delta f(x) = E^{\frac{1}{2}}f(x) - E^{-\frac{1}{2}}f(x)$$

$$\Rightarrow \delta f(x) = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)f(x)$$

$$\Rightarrow \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$\Rightarrow \delta^2 \equiv \left(E^{\frac{1}{2}}\right)^2 + \left(E^{-\frac{1}{2}}\right)^2 - 2 \cdot E^{\frac{1}{2}} \cdot E^{-\frac{1}{2}} \quad [\because (A - B)^2 = A^2 + B^2 - 2A \cdot B]$$

$$\Rightarrow \delta^2 \equiv E + E^{-1} - 2$$

(d) Relationship between μ and E :

We know that $\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$

$$\Rightarrow \mu f(x) = \frac{1}{2}\left[E^{\frac{1}{2}}f(x) + E^{-\frac{1}{2}}f(x)\right]$$

$$\Rightarrow \mu f(x) = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]f(x)$$

$$\Rightarrow \mu \equiv \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]$$

$$\Rightarrow \mu^2 \equiv \frac{1}{4}\left[\left(E^{\frac{1}{2}}\right)^2 + \left(E^{-\frac{1}{2}}\right)^2 + 2 \cdot E^{\frac{1}{2}} \cdot E^{-\frac{1}{2}}\right] \quad [\because (A + B)^2 = A^2 + B^2 + 2A \cdot B]$$

$$\Rightarrow \mu^2 \equiv \frac{1}{4}[E + E^{-1} + 2]$$

(e) Relationship between μ and δ :

We know that $\mu^2 \equiv \frac{1}{4}[E + E^{-1} + 2]$

$$\Rightarrow \mu^2 \equiv \frac{1}{4}[E + E^{-1} - 2 + 4]$$

$$\Rightarrow \mu^2 \equiv \frac{1}{4}[(E + E^{-1} - 2) + 4]$$

$$\Rightarrow \mu^2 \equiv \frac{1}{4}[\delta^2 + 4] \Rightarrow \mu^2 \equiv 1 + \frac{1}{4}\delta^2$$

Example 3.3.1: Prove the given relation between difference operators:

(i) $\delta^2 \equiv E - 2 + E^{-1}$

(ii) $\mu\delta \equiv \frac{1}{2}(\Delta + \nabla)$

Solution: We Know that:

• $\Delta \equiv E - 1$	$\Rightarrow$	$E \equiv 1 + \Delta$
• $\nabla \equiv 1 - E^{-1}$	$\Rightarrow$	$E^{-1} \equiv 1 - \nabla$
• $\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$	$\Rightarrow$	$\delta^2 \equiv E + E^{-1} - 2$
• $\mu \equiv \frac{1}{2} \left[E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right]$	$\Rightarrow$	$\mu^2 \equiv \frac{1}{4} [E + E^{-1} + 2]$

Proof-(i) We know that:

$$\begin{aligned} \delta &\equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}} \\ \Rightarrow \delta^2 &\equiv \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right)^2 \\ \Rightarrow \delta^2 &\equiv \left(E^{\frac{1}{2}} \right)^2 - 2 \cdot E^{\frac{1}{2}} \cdot E^{-\frac{1}{2}} + \left(E^{-\frac{1}{2}} \right)^2 \\ \Rightarrow \delta^2 &\equiv E - 2 + E^{-1} \end{aligned}$$

Proof-(ii) We start as follows:

$$\begin{aligned} \text{L.H.S} \equiv \mu\delta &\equiv \frac{1}{2} \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right) \cdot \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right) & [\because (A+B)(A-B) = A^2 - B^2] \\ \Rightarrow &\equiv \frac{1}{2} \left[\left(E^{\frac{1}{2}} \right)^2 - \left(E^{-\frac{1}{2}} \right)^2 \right] \\ \Rightarrow &\equiv \frac{1}{2} [E - E^{-1}] \\ \Rightarrow &\equiv \frac{1}{2} [1 + \Delta - E^{-1}] & [\because E \equiv 1 + \Delta] \\ \Rightarrow &\equiv \frac{1}{2} [\Delta + 1 - E^{-1}] & [\because \nabla \equiv 1 - E^{-1}] \\ \Rightarrow &\equiv \frac{1}{2} [\Delta + \nabla] \equiv \text{R.H.S} \end{aligned}$$

Example 3.3.2: Prove that: $E \equiv e^{hD}$, where D is differentiation operator.

Solution: We Know that:

$$Ef(x) = f(x+h) \quad [\text{Using Taylors Series}]$$

$$\begin{aligned} \Rightarrow Ef(x) &= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots \\ \Rightarrow Ef(x) &= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \dots + \infty \\ \Rightarrow Ef(x) &= \left[1 + (hD) + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \dots + \infty \right] f(x) \\ \Rightarrow Ef(x) &= e^{hD} f(x) \Rightarrow E \equiv e^{hD} \end{aligned}$$

3.4 Newton's Forward Difference Interpolation Formula

When step-size h has same value for all intervals as follows:

Let $y = f(x)$, $x_0 \leq x \leq x_n$ be a function then we can write:

x	x_0	$x_0 + h$	$x_0 + 2h$	$x_0 + 3h$	$\dots$	$x_0 + nh$
$y = f(x)$	$f(x_0)$	$f(x_0 + h)$	$f(x_0 + 2h)$	$f(x_0 + 3h)$	$\dots$	$f(x_0 + nh)$

Suppose we wish to evaluate $f(x)$ for $x = x_0 + sh$, $\Rightarrow s = \frac{x-x_0}{h}$, we know that:

$$f(x) = f(x_0 + sh) = E^s f(x_0) = (1 + \Delta)^s f(x_0) \quad [\because E^s f(x) = f(x + sh)] \text{ \& (Binomial Theorem)}$$

$$= [1 + s\Delta + \frac{s(s-1)}{2!} \Delta^2 + \frac{s(s-1)(s-2)}{3!} \Delta^3 + \dots + \frac{s(s-1)(s-2)\dots(s-n+1)}{n!} \Delta^n + \dots] f(x_0)$$

$$P_n(x) = f(x_0 + sh) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!} \Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!} \Delta^3 f(x_0) + \dots$$

$$+ \frac{s(s-1)(s-2)\dots(s-n+1)}{n!} \Delta^n f(x_0)$$

$$\Rightarrow P_n(x) = f(x_0 + sh) = f(x_0) + \sum_{k=1}^n \binom{s}{k} \Delta^k f(x_0)$$

This is **Newton's Forward Difference** Interpolation Polynomial of degree n

Note:

(a) Newton's Forward Difference Interpolation Polynomial of Degree 1: (Taking $n = 1$)

$$P_1(x) = f(x_0 + sh) = f(x_0) + s\Delta f(x_0)$$

(b) Newton's Forward Difference Interpolation Polynomial of Degree 2: (Taking $n = 2$)

$$P_2(x) = f(x_0 + sh) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!} \Delta^2 f(x_0)$$

(c) Newton's Forward Difference Interpolation Polynomial of Degree 3: (Taking $n = 3$)

$$P_3(x) = f(x_0 + sh) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!} \Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!} \Delta^3 f(x_0)$$

Example 3.4.1: Use Newton's Forward Difference Interpolation Formula to construct interpolating polynomials of degree 1, 2 and 3 for the following data. Approximate the specified value using each of the polynomials: $f(0.5)$ if $f(0) = 1$, $f(1) = 2$, $f(2) = 1$, $f(3) = 10$

Solution: Let us construct forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1	1	-2	12
1	2	-1	10	
2	1	9		
3	10			

Here: $h = 1$, $x_0 = 0$ and so $x = x_0 + sh \Rightarrow x = 0 + s.1 = s$

Newton's Forward Difference Interpolation Polynomial of Degree 1

Taking, $h = 1$, $x_0 = 0$ and so $x = s$ in $P_1(x)$ we get:

$$\begin{aligned}
 P_1(x) &= f(x_0 + sh) = f(x_0) + s\Delta f(x_0) \\
 \Rightarrow P_1(x) &= f(0 + x.1) = f(0) + x\Delta f(0) = 1 + x.1 \\
 \Rightarrow P_1(x) &= 1 + x \quad \text{(This is Polynomial of Degree 1)} \\
 \Rightarrow P_1(0.5) &= 1 + 0.5 = 1.5
 \end{aligned}$$

Newton's Forward Difference Interpolation Polynomial of Degree 2

Taking, $h = 1$, $x_0 = 0$ and $x = s$ in $P_2(x)$ we get:

$$\begin{aligned}
 P_2(x) &= f(x_0 + sh) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) \\
 \Rightarrow P_2(x) &= f(0 + x.1) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0) \\
 \Rightarrow P_2(x) &= 1 + x.1 + \frac{x(x-1)}{2!}(-2) \\
 \Rightarrow P_2(x) &= 1 + x - x(x-1) \\
 \Rightarrow P_2(x) &= 1 + x - x^2 + x \\
 \Rightarrow P_2(x) &= 1 + 2x - x^2 \quad \text{(This is Polynomial of Degree 2)} \\
 \Rightarrow P_2(0.5) &= 1 + 2(0.5) - (0.5)^2 \\
 \Rightarrow P_2(x) &= 1.75
 \end{aligned}$$

Newton's Forward Difference Interpolation Polynomial of Degree 3

Taking, $h = 1$, $x_0 = 0$ and $x = s$ in $P_3(x)$ we get:

$$\begin{aligned}
 P_3(x) &= f(x_0 + sh) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0) \\
 \Rightarrow P_3(x) &= f(0 + x.1) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(0) \\
 \Rightarrow P_3(x) &= 1 + x.1 + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12) \\
 \Rightarrow P_3(x) &= 1 + x - x(x-1) + 2x(x-1)(x-2) \\
 \Rightarrow P_3(x) &= 1 + x - x^2 + x + 2x(x^2 - 3x + 2) \\
 \Rightarrow P_3(x) &= 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x \\
 \Rightarrow P_3(x) &= 1 + 6x - 7x^2 + 2x^3 \quad \text{(This is Polynomial of Degree 3)} \\
 \Rightarrow P_3(0.5) &= 1 + 6(0.5) - 7(0.5)^2 + 2(0.5)^3 \\
 \Rightarrow P_3(0.5) &= 1 + 6(0.5) - 7(0.5)^2 + 2(0.5)^3 = 2.5
 \end{aligned}$$

3.5 Newton's Backward Difference Interpolation Formula

When step-size h has same value for all intervals as follows:

Let $y = f(x)$, $x_0 \leq x \leq x_n$ be a function then we can write:

x	x_0	$x_0 + h$	$x_0 + 2h$	$x_0 + 3h$	$\dots$	$x_0 + nh$
$y = f(x)$	$f(x_0)$	$f(x_0 + h)$	$f(x_0 + 2h)$	$f(x_0 + 3h)$	$\dots$	$f(x_0 + nh)$

Suppose, we wish to evaluate $f(x)$ for $x = x_n + sh \Rightarrow s = \frac{x - x_n}{h}$, we know that:

$$f(x) = f(x_n + sh) = E^s f(x_n) = (E^{-1})^{-s} f(x_n) = (1 - \nabla)^{-s} f(x_n) \quad [\because E^{-s} f(x) = f(x - sh)]$$

$$= [1 + s\nabla + \frac{s(s+1)}{2!} \nabla^2 + \frac{s(s+1)(s+2)}{3!} \nabla^3 + \dots + \frac{s(s+1)(s+2)\dots(s+n-1)}{n!} \nabla^n + \dots] f(x_n)$$

$$\Rightarrow P_n(x) = f(x_n + sh) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!} \nabla^2 f(x_n) + \frac{s(s+1)(s+2)}{3!} \nabla^3 f(x_n) + \dots$$

$$+ \frac{s(s+1)(s+2)\dots(s+n-1)}{n!} \nabla^n f(x_n)$$

$$\Rightarrow P_n(x) = f(x_n + sh) = f(x_n) + \sum_{k=1}^n (-1)^k \binom{-s}{k} \nabla^k f(x_n)$$

This is **Newton's Backward Difference** Interpolation Formula.

Binomial Theorem:

$$(1 + a)^s = [1 + sa + \frac{s(s-1)}{2!} a^2 + \frac{s(s-1)(s-2)}{3!} a^3 + \dots + \frac{s(s-1)(s-2)\dots(s-n+1)}{n!} a^n + \dots]$$

$$(1 - a)^{-s} = [1 + sa + \frac{s(s+1)}{2!} a^2 + \frac{s(s+1)(s+2)}{3!} a^3 + \dots + \frac{s(s+1)(s+2)\dots(s+n-1)}{n!} a^n + \dots]$$

Example 3.5.1: The population of a town is given. Estimate the population for the year 1975.

Years: x	1941	1951	1961	1971	1981	1991
Population(in million): $f(x)$	20	24	29	36	46	51

Solution: Since the value to be interpolated is at the end of the table therefore we use Newton's Backward Difference Interpolation Formula. Let us construct backward difference table:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
1941	20					
1951	24	4				
1961	29	5	1			
1971	36	7	2	1		
1981	46	10	3	1	0	
1991	51	5	-5	-8	-9	-9

Here, $h = 10$, $x_n = 1991$, $x = 1975$ then $x = x_n + sh \Rightarrow s = \frac{x-x_n}{h} = \frac{1975-1991}{10} = -1.6$

Applying **Newton's Backward Difference** Interpolation Formula of degree 5:

$$\Rightarrow P_5(x) = f(x_5 + sh) = f(x_5) + s\nabla f(x_5) + \frac{s(s+1)}{2!}\nabla^2 f(x_5) + \frac{s(s+1)(s+2)}{3!}\nabla^3 f(x_5) + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 f(x_5) + \frac{s(s+1)(s+2)(s+3)(s+4)}{5!}\nabla^5 f(x_5)$$

$$\Rightarrow P_5(1975) = 51 + (-1.6)5 + \frac{-1.6(-1.6+1)}{2}(-5) + \frac{-1.6(-1.6+1)(-1.6+2)}{6}(-8) + \frac{-1.6(-1.6+1)(-1.6+2)(-1.6+3)}{24}(-9) + \frac{-1.6(-1.6+1)(-1.6+2)(-1.6+3)(-1.6+4)}{120}(-9)$$

$$\Rightarrow P_5(1975) = 39.789632 \text{ Millions}$$

3.6 Lagrange Interpolating Polynomial of Degree n:

When step-size h may not necessarily having same value for all intervals:

Let $y = f(x)$, $x_0 \leq x \leq x_n$ be a function given by the following table:

x	x_0	x_1	x_2	x_3	$\cdots$	x_n
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\cdots$	$f(x_n)$
y	y_0	y_1	y_2	y_3		y_n

We can represent $f(x)$ by a polynomial of degree n as follows:

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \dots + A_nx^n \quad \dots (1)$$

Also, more conveniently, in the following form:

$$f(x) = a_0(\mathbf{x} - x_1)(\mathbf{x} - x_2)(\mathbf{x} - x_3) \dots (\mathbf{x} - x_n) + a_1(\mathbf{x} - x_0)(\mathbf{x} - x_2)(\mathbf{x} - x_3) \dots (\mathbf{x} - x_n) \\ + a_2(\mathbf{x} - x_0)(\mathbf{x} - x_1)(\mathbf{x} - x_3) \dots (\mathbf{x} - x_n) + \dots + a_n(\mathbf{x} - x_0)(\mathbf{x} - x_1)(\mathbf{x} - x_2) \dots (\mathbf{x} - x_{n-1}) \dots (2)$$

Here, $a_0, a_1, a_2, \dots, a_n$ are chosen such that the function $f(x)$ satisfies the data **(given in the above table)**.

From equation ... (2) we have-

$$\begin{array}{lll} f(x_0) = a_0(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \dots (x_0 - x_n) & \Rightarrow & a_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \dots (x_0 - x_n)} \\ f(x_1) = a_1(x_1 - x_0)(x_1 - x_2)(x_1 - x_3) \dots (x_1 - x_n) & \Rightarrow & a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3) \dots (x_1 - x_n)} \\ \dots & & \dots \\ f(x_n) = a_n(x_n - x_0)(x_n - x_1)(x_n - x_2) \dots (x_n - x_{n-1}) & \Rightarrow & a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)(x_n - x_2) \dots (x_n - x_{n-1})} \end{array}$$

Now substitute the value of $a_0, a_1, a_2, \dots, a_n$ into equation ... (2), we get:

$$f(\mathbf{x}) = \frac{(x-x_1)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\dots(x_0-x_n)} \cdot f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)} \cdot f(x_1) \\ + \frac{(x-x_0)(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)} \cdot f(x_2) + \dots + \frac{(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})} \cdot f(x_n)$$

$$\Rightarrow f(\mathbf{x}) = L_0(\mathbf{x}) \cdot f(x_0) + L_1(\mathbf{x}) \cdot f(x_1) + L_2(\mathbf{x}) \cdot f(x_2) + \dots + L_n(\mathbf{x}) \cdot f(x_n)$$

$$\Rightarrow f(\mathbf{x}) = \sum_{k=0}^n L_k(\mathbf{x}) \cdot f(\mathbf{x}_k) \quad \dots (3)$$

This is Lagrange Interpolating Polynomial of Degree n

(a) Lagrange Interpolating Polynomial of Degree 1:

Since,

x	x_0	x_1
$y = f(x)$	$f(x_0)$	$f(x_1)$
y	y_0	y_1

Taking $n = 1$ in equation-(3), we get:

$$f(x) = \sum_{k=0}^1 L_k(x) \cdot f(x_k)$$

$$\Rightarrow f(x) = L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1)$$

$$\Rightarrow f(x) = \frac{(x-x_1)}{(x_0-x_1)} \cdot f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} \cdot f(x_1)$$

Example 3.6.1: If $f(-2) = 1$ and $f(-1) = 4$, then the **Lagrange's interpolating polynomial** of degree **one** is given by:

(a) $7x - 3$

(b) $3x + 7$

(c) $3x - 7$

(d) $7x + 3$

Solution: Given that

	x_0	x_1
x	-2	-1
$y = f(x)$	1	4
	$f(x_0)$	$f(x_1)$

The **Lagrange's interpolating polynomial** of degree **one** is given by:

$$f(x) = L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1)$$

$$\Rightarrow f(x) = \frac{(x-x_1)}{(x_0-x_1)} \cdot f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} \cdot f(x_1)$$

$$\Rightarrow f(x) = \frac{(x-(-1))}{(-2-(-1))} \cdot (1) + \frac{(x-(-2))}{(-1-(-2))} \cdot (4)$$

$$\Rightarrow f(x) = \frac{(x+1)}{(-1)} \cdot (1) + \frac{(x+2)}{(1)} \cdot (4)$$

$$\Rightarrow f(x) = -x - 1 + 4x + 8$$

$$\Rightarrow f(x) = 3x + 7$$

Example 3.6.2: Determine the linear Lagrange Interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

Solution: Given that

	x_0	x_1
x	2	5
$y = f(x)$	4	1
	$f(x_0)$	$f(x_1)$

The **Lagrange's interpolating polynomial** of degree **one** is given by:

$$\begin{aligned}
 f(x) &= L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1) \\
 \Rightarrow f(x) &= \frac{(x-x_1)}{(x_0-x_1)} \cdot f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} \cdot f(x_1) \\
 \Rightarrow f(x) &= \frac{(x-5)}{(2-5)} \cdot (4) + \frac{(x-2)}{(5-2)} \cdot (1) \\
 \Rightarrow f(x) &= \frac{(x-5)}{(-3)} \cdot (4) + \frac{(x-2)}{(3)} \cdot (1) \\
 \Rightarrow f(x) &= -\frac{4x}{3} + \frac{20}{3} + \frac{x}{3} - \frac{2}{3} = -x + 6
 \end{aligned}$$

Example 3.6.3:

(a) Use the numbers (called *nodes*) $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$.

(b) Use this polynomial to approximate $f(3) = 1/3$.

Solution: Given that

	x_0	x_1	x_2
x	2	2.75	4
$y = f(x) = 1/x$	1/2	4/11	1/4
	$f(x_0)$	$f(x_1)$	$f(x_2)$

(a) The **Lagrange's interpolating polynomial** of degree **two** is given by:

$$\begin{aligned}
 f(x) &= L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1) + L_2(x) \cdot f(x_2) \\
 \Rightarrow f(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot f(x_2) \\
 \Rightarrow f(x) &= \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} \cdot \left(\frac{1}{2}\right) + \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} \cdot \left(\frac{4}{11}\right) + \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} \cdot \left(\frac{1}{4}\right) \\
 \Rightarrow f(x) &= \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75) \\
 \Rightarrow f(x) &= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}
 \end{aligned}$$

(b) Taking $x = 3$ in **Lagrange's interpolating polynomial** of degree **two** obtained in (a), we get

$$\begin{aligned}
 f(x) &= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44} \\
 \Rightarrow f(3) &= \frac{1}{22}(3)^2 - \frac{35}{88}(3) + \frac{49}{44} \\
 \Rightarrow f(3) &= \frac{29}{88} = 0.329545455
 \end{aligned}$$

Example 3.6.4: Explain and illustrate Lagrange's Interpolation Method to approximate $f(2)$ for the given table:

x	1	3	4
$f(x)$	7	37	76

Solution: We have

	x_0	x_1	x_2
x	1	3	4
$f(x)$	7	37	76
	$f(x_0)$	$f(x_1)$	$f(x_2)$

Using Lagrange Interpolating Polynomial-

$$f(x) = L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1) + L_2(x) \cdot f(x_2)$$

$$\Rightarrow f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot f(x_2)$$

$$\Rightarrow f(2) = \frac{(2-3)(2-4)}{(1-3)(1-4)} \cdot (7) + \frac{(2-1)(2-4)}{(3-1)(3-4)} \cdot (37) + \frac{(2-1)(2-3)}{(4-1)(4-3)} \cdot (76)$$

$$\Rightarrow f(2) = \frac{1}{3} \cdot (7) + \frac{-2}{-2} \cdot (37) + \frac{-1}{3} \cdot (76)$$

$$\Rightarrow f(2) = 14$$

Example 3.6.5: Use Lagrange Interpolating Polynomial to approximate $f(155)$ if

$$f(150) = 12.247, f(152) = 12.329, f(154) = 12.410, f(156) = 12.490$$

Solution: We have

	x_0	x_1	x_2	x_3
x	150	152	154	156
$f(x)$	12.247	12.329	12.410	12.490
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Using Lagrange Interpolating Polynomial-

$$f(x) \approx P(x) = L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1) + L_2(x) \cdot f(x_2) + L_3(x) \cdot f(x_3)$$

$$\Rightarrow f(155) \approx \frac{(155-152)(155-154)(155-156)}{(150-152)(150-154)(150-156)} \cdot (12.247) + \frac{(155-150)(155-154)(155-156)}{(152-150)(152-154)(152-156)} \cdot (12.329)$$

$$+ \frac{(155-150)(155-152)(155-156)}{(154-150)(154-152)(154-156)} \cdot (12.410) + \frac{(155-150)(155-152)(155-154)}{(156-150)(156-152)(156-154)} \cdot (12.490)$$

$$\Rightarrow f(155) \approx 0.7654375 - 3.8528125 + 11.634375 + 3.903125$$

$$\Rightarrow f(155) \approx 12.450125$$

Example 3.6.6: Use **Lagrange Interpolating Formula** to find **Polynomial $f(x)$** and hence approximate **$f'(3)$** if, **$f(0) = 2$, $f(1) = 3$, $f(2) = 12$, $f(5) = 147$**

Solution: We have

	x_0	x_1	x_2	x_3
x	0	1	2	5
$f(x)$	2	3	12	147
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

We know that the **Lagrange Interpolating Polynomial of degree 3** is given by:

$$f(x) = L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1) + L_2(x) \cdot f(x_2) + L_3(x) \cdot f(x_3)$$

$$\Rightarrow f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot f(x_1) \\ + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot f(x_3)$$

$$\Rightarrow f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \cdot (2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \cdot (3) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \cdot (12) \\ + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \cdot (147)$$

$$\Rightarrow f(x) = \frac{(x^3-8x^2+17x-10)}{-10} \cdot (2) + \frac{(x^3-7x^2+10x)}{4} \cdot (3) + \frac{(x^3-6x^2+5x)}{-6} \cdot (12) + \frac{(x^3-3x^2+2x)}{60} \cdot (147)$$

$$\Rightarrow f(x) = x^3 \left(\frac{2}{-10} + \frac{3}{4} + \frac{12}{-6} + \frac{147}{60} \right) + x^2 \left(\frac{-16}{-10} + \frac{-21}{4} + \frac{-72}{-6} + \frac{-147}{20} \right) + x \left(\frac{34}{-10} + \frac{30}{4} + \frac{60}{-6} + \frac{147}{30} \right) + 1 \left(\frac{-20}{-10} \right)$$

$$\Rightarrow f(x) = x^3 + x^2 - x + 2$$

$$\Rightarrow f'(x) = 3x^2 + 2x - 1$$

$$\Rightarrow f'(3) = 27 + 6 - 1 = 32$$

3.7 Newton's Divided Difference Interpolation Formula:

When step-size h may not necessarily having same value for all intervals.

Let $y = f(x)$, $x_0 \leq x \leq x_n$ be a function then we can write:

x	x_0	x_1	x_2	x_3	$\dots$	x_n
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\dots$	$f(x_n)$

We first need to introduce the divided-difference notation, which are as follows:

(a) Zeroth Divided Difference of the Function $f(x)$:

$$f[x_i] = f(x_i) \text{ for each } i = 0, 1, \dots, n$$

$$f[x_0] = f(x_0), \quad f[x_1] = f(x_1), \quad f[x_2] = f(x_2), \dots, f[x_n] = f(x_n)$$

(b) First Divided Difference of the Function $f(x)$:

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}, \quad \text{for each } i = 0, 1, \dots, n-1$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}, \quad f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}, \quad \dots, \quad f[x_{n-1}, x_n] = \frac{f[x_n] - f[x_{n-1}]}{x_n - x_{n-1}}$$

(c) Second Divided Difference of the Function $f(x)$:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}, \quad \text{for each } i = 0, 1, \dots, n-2$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}, \quad f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}, \dots,$$

$$f[x_{n-2}, x_{n-1}, x_n] = \frac{f[x_{n-1}, x_n] - f[x_{n-2}, x_{n-1}]}{x_n - x_{n-2}}$$

(d) n^{th} Divided Difference of the Function $f(x)$:

$$f[x_0, x_1, x_2, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0} \quad (\text{Single value})$$

(e) Divided Difference Table:

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
x_1	$f[x_1]$	$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_2	$f[x_2]$	$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$	$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
x_3	$f[x_3]$	$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$	$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
x_4	$f[x_4]$	$f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3}$	$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
x_5	$f[x_5]$	$f[x_4, x_5] = \frac{f[x_5] - f[x_4]}{x_5 - x_4}$	$f[x_3, x_4, x_5] = \frac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	$f[x_2, x_3, x_4, x_5] = \frac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$

(f) Newton's Divided Difference Interpolation Formula:

We know that- $f[x, x_0] = \frac{f(x) - f(x_0)}{x - x_0} \Rightarrow f(x) = f[x_0] + (x - x_0)f[x, x_0] \dots \dots \dots (1)$

From second divided difference, we can write:

$$f[x, x_0, x_1] = \frac{f[x, x_0] - f[x_0, x_1]}{x - x_1} \Rightarrow f[x, x_0] = f[x_0, x_1] + (x - x_1)f[x, x_0, x_1] \dots \dots \dots (2)$$

From (1) and (2), we have-

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x, x_0, x_1] \dots \dots \dots (3)$$

From third divided difference, we can write:

$$f[x, x_0, x_1, x_2] = \frac{f[x, x_0, x_1] - f[x_0, x_1, x_2]}{x - x_2}$$

$$\Rightarrow f[x, x_0, x_1] = f[x_0, x_1, x_2] + (x - x_2)f[x, x_0, x_1, x_2] \dots \dots \dots (4)$$

From (3) and (4), we have-

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x, x_0, x_1, x_2]$$

Similarly, we can write:

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x, x_0, x_1, x_2, x_3]$$

Continuing in this manner, we can get **Newton's Divided Difference Interpolation Formula of degree n** as follows:

$$f(x) = f[x_0] + (x - x_0) \cdot f[x_0, x_1] + (x - x_0)(x - x_1) \cdot f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) \cdot f[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3) \cdot f[x_0, x_1, x_2, x_3, x_4] + (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1}) \cdot f[x_0, x_1, x_2, \dots, x_n]$$

OR

$$f(x) \approx P_n(x) = f[x_0] + \sum_{k=1}^n (x - x_0)(x - x_1) \cdots (x - x_{k-1}) \cdot f[x_0, x_1, \dots, x_k]$$

Example 3.7.1: Construct the **Newton's Divided Differences Table** for the following data:

x	0	2	3	6
$y = f(x)$	648	704	729	792

Solution: Let us Construct Newton's Divided Difference (DD) Table:

	x	$f[x]$	1 st DD	2 nd DD	3 rd DD
x_0	0	648			
			$\frac{704 - 648}{2 - 0} = 28$		
x_1	2	704		$\frac{25 - 28}{3 - 0} = -1$	
			$\frac{729 - 704}{3 - 2} = 25$		$\frac{-1 + 1}{6 - 0} = 0$
x_2	3	729		$\frac{21 - 25}{6 - 2} = -1$	
			$\frac{792 - 729}{6 - 3} = 21$		
x_3	6	792			

Example 3.7.2: Use **Newton's Divided Difference Interpolation Formula** to approximate $f(9)$ if $f(5) = 150$, $f(7) = 392$, $f(11) = 1452$, $f(13) = 2366$ and $f(17) = 5202$

Solution: Construct Newton's Divided Difference (DD) Table:

	x	$f[x]$	Newton's 1 st DD	Newton's 2 nd DD	Newton's 3 rd DD	Newton's 4 th DD
x_0	5	150				
			$\frac{392 - 150}{7 - 5} = 121$			
x_1	7	392		$\frac{265 - 121}{11 - 5} = 24$		
			$\frac{1452 - 392}{11 - 7} = 265$		$\frac{32 - 24}{13 - 5} = 1$	
x_2	11	1452		$\frac{457 - 265}{13 - 7} = 32$		$\frac{1 - 1}{17 - 5} = 0$
			$\frac{2366 - 1452}{13 - 11} = 457$		$\frac{42 - 32}{17 - 7} = 1$	
x_3	13	2366		$\frac{709 - 457}{17 - 11} = 42$		
			$\frac{5202 - 2366}{17 - 13} = 709$			
x_4	17	5202				

From Newton's divided difference (DD) table, we have:

$$f[x_0] = 150, f[x_0, x_1] = 121, f[x_0, x_1, x_2] = 24, f[x_0, x_1, x_2, x_3] = 1 \text{ and } f[x_0, x_1, x_2, x_3, x_4] = 0$$

Substitute these in the **Newton's divided difference interpolation formula**, we get-

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3, x_4]$$

$$\Rightarrow f(x) = 150 + (x - 5)121 + (x - 5)(x - 7)24 + (x - 5)(x - 7)(x - 11)1 + (x - 5)(x - 7)(x - 11)(x - 13)0$$

$$\Rightarrow f(9) = 150 + (9 - 5)121 + (9 - 5)(9 - 7)24 + (9 - 5)(9 - 7)(9 - 11)1 + (9 - 5)(9 - 7)(9 - 11)(9 - 13)0$$

$$\Rightarrow f(9) = 810$$

Example 3.7.3: Use **Newton Divided Difference Interpolating Formula** to find **Polynomial $f(x)$** and hence approximate **$f'(5)$** if

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution: Let us Construct **Newton's divided difference (DD) table:**

	x	$f[x]$	Newton's 1 st DD	Newton's 2 nd DD	Newton's 3 rd DD	Newton's 4 th DD	Newton's 5 th DD
x_0	4	48					
			$\frac{100 - 48}{5 - 4} = 52$				
x_1	5	100		$\frac{97 - 52}{7 - 4} = 15$			
			$\frac{294 - 100}{7 - 5} = 97$		$\frac{21 - 15}{10 - 4} = 1$		
x_2	7	294		$\frac{202 - 97}{10 - 5} = 21$		$\frac{1 - 1}{11 - 4} = 0$	
			$\frac{900 - 294}{10 - 7} = 202$		$\frac{27 - 21}{11 - 5} = 1$		$\frac{0 - 0}{13 - 4} = 0$
x_3	10	900		$\frac{310 - 202}{11 - 7} = 27$		$\frac{1 - 1}{13 - 5} = 0$	
			$\frac{1210 - 900}{11 - 10} = 310$		$\frac{33 - 27}{13 - 7} = 1$		
x_4	11	1210		$\frac{409 - 310}{13 - 10} = 33$			
			$\frac{2028 - 1210}{13 - 11} = 409$				
x_5	13	2028					

From Newton's divided difference (DD) table, we have:

$$f[x_0] = 48, \quad f[x_0, x_1] = 52, \quad f[x_0, x_1, x_2] = 15, \quad f[x_0, x_1, x_2, x_3] = 1 \text{ and } f[x_0, x_1, x_2, x_3, x_4] = 0$$

Substitute in Newton's divided difference interpolation formula:

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

$$+ (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3, x_4]$$

$$\Rightarrow f(x) = 48 + (x - 4)52 + (x - 4)(x - 5)15 + (x - 4)(x - 5)(x - 7)1 + 0 + 0$$

$$\Rightarrow f(x) = 48 + (x - 4)52 + (x^2 - 9x + 20)15 + (x^3 - 16x^2 + 83x - 140)$$

$$\Rightarrow f(x) = x^3 - x^2 \quad \Rightarrow \quad f'(x) = 3x^2 - 2x \quad \Rightarrow \quad f'(5) = 75 - 10 = 65$$

Example 3.7.4: Use **Newton Divided Difference Interpolating Formula** to show that the **polynomial** interpolating the following data has **degree 3**.

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution: Construct **Newton's Divided Difference (DD) Table**:

	x	$f[x]$	Newton's 1 st DD	Newton's 2 nd DD	Newton's 3 rd DD	Newton's 4 th DD
x_0	4	48				
			$\frac{100 - 48}{5 - 4} = 52$			
x_1	5	100		$\frac{97 - 52}{7 - 4} = 15$		
			$\frac{294 - 100}{7 - 5} = 97$		$\frac{21 - 15}{10 - 4} = 1$	
x_2	7	294		$\frac{202 - 97}{10 - 5} = 21$		$\frac{1 - 1}{11 - 4} = 0$
			$\frac{900 - 294}{10 - 7} = 202$		$\frac{27 - 21}{11 - 5} = 1$	
x_3	10	900		$\frac{310 - 202}{11 - 7} = 27$		$\frac{1 - 1}{13 - 5} = 0$
			$\frac{1210 - 900}{11 - 10} = 310$		$\frac{33 - 27}{13 - 7} = 1$	
x_4	11	1210		$\frac{409 - 310}{13 - 10} = 33$		
			$\frac{2028 - 1210}{13 - 11} = 409$			
x_5	13	2028				

From Newton's divided difference (DD) table, we have:

$$f[x_0] = 48, \quad f[x_0, x_1] = 52, \quad f[x_0, x_1, x_2] = 15, \quad f[x_0, x_1, x_2, x_3] = 1 \text{ and } f[x_0, x_1, x_2, x_3, x_4] = 0$$

Substitute in Newton's divided difference interpolation formula:

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

$$+ (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3, x_4]$$

$$\Rightarrow f(x) = 48 + (x - 4)52 + (x - 4)(x - 5)15 + (x - 4)(x - 5)(x - 7)1 + 0 + 0$$

$$\Rightarrow f(x) = 48 + (x - 4)52 + (x^2 - 9x + 20)15 + (x^3 - 16x^2 + 83x - 140)$$

$$\Rightarrow f(x) = x^3 - x^2$$

3.8 Inverse Interpolation:

Let $y = f(x)$, $x_0 \leq x \leq x_n$ be a function then we can write:

x	x_0	x_1	x_2	x_3	$\dots$	x_n
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\dots$	$f(x_n)$
y	y_0	y_1	y_2	y_3		y_n

Definition: The process of estimating the value of x for the value of y not in the table is called **Inverse Interpolation**.

For the sake of simplicity, Let $x = F(y)$, $y_0 \leq y \leq y_n$ be a function then we can write:

$x = F(y)$	$F(y_0)$	$F(y_1)$	$F(y_2)$	$F(y_3)$	$\dots$	$F(y_n)$
x	x_0	x_1	x_2	x_3	$\dots$	x_n
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\dots$	$f(x_n)$
y	y_0	y_1	y_2	y_3		y_n

Working Rule: When the values of x are unevenly spaced then we can use the following Inverse Interpolation Formula:

(a) Lagrange Inverse Interpolating Formula:

$$x = F(y) = \sum_{k=0}^n L_k(y) \cdot F(y_k)$$

$$\Rightarrow x = F(y) = L_0(y) \cdot F(y_0) + L_1(y) \cdot F(y_1) + L_2(y) \cdot F(y_2) + \dots + L_n(y) \cdot F(y_n)$$

$$\Rightarrow x = F(y) = \frac{(y-y_1)(y-y_2)(y-y_3)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\dots(y_0-y_n)} \cdot F(y_0) + \frac{(y-y_0)(y-y_2)(y-y_3)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)\dots(y_1-y_n)} \cdot F(y_1) \\ + \frac{(y-y_0)(y-y_1)(y-y_3)\dots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\dots(y_2-y_n)} \cdot F(y_2) + \dots + \frac{(y-y_0)(y-y_1)(y-y_2)\dots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)\dots(y_n-y_{n-1})} \cdot F(y_n)$$

This is **Lagrange Interpolating Polynomial** of Degree n for **Inverse Interpolation**

(b) Newton's Divided Difference Inverse Interpolating Formula:

$$x = F(y) = F[y_0] + (y - y_0) \cdot F[y_0, y_1] + (y - y_0)(y - y_1) \cdot F[y_0, y_1, y_2] \\ + (y - y_0)(y - y_1)(y - y_2) \cdot F[y_0, y_1, y_2, y_3] \\ + (y - y_0)(y - y_1)(y - y_2)(y - y_3) \cdot F[y_0, y_1, y_2, y_3, y_4] \\ + \dots \\ + (y - y_0)(y - y_1)(y - y_2) \dots (y - y_{n-1}) \cdot F[y_0, y_1, y_2, \dots, y_n]$$

This is **Newton's Divided Difference Interpolating Polynomial** of Degree n for **Inverse Interpolation**

Example 3.8.1: Using **Lagrange Inverse Interpolating Formula**, find the value of x for $f(x) = 0.390$, If $f(20) = 0.342$, $f(25) = 0.435$, $f(30) = 0.5$ and $f(35) = 0.65$

Solution: Rewriting in tabular form-

$x = F(y)$	$F(y_0)$	$F(y_1)$	$F(y_2)$	$F(y_3)$
x	20	25	30	35
$y = f(x)$	0.342	0.423	0.5	0.65
y	y_0	y_1	y_2	y_3

Using **Lagrange Inverse Interpolating Formula**

$$F(y) = L_0(y) \cdot F(y_0) + L_1(y) \cdot F(y_1) + L_2(y) \cdot F(y_2) + L_3(y) \cdot F(y_3)$$

$$\Rightarrow x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} \cdot F(y_0) + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} \cdot F(y_1)$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} \cdot F(y_2) + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} \cdot F(y_3)$$

$$\Rightarrow x = \frac{(0.39-0.423)(0.39-0.5)(0.39-0.65)}{(0.342-0.423)(0.342-0.5)(0.342-0.65)} \cdot (20) + \frac{(0.39-0.342)(0.39-0.5)(0.39-0.65)}{(0.423-0.342)(0.423-0.5)(0.423-0.65)} \cdot (25)$$

$$+ \frac{(0.39-0.342)(0.39-0.423)(0.39-0.65)}{(0.5-0.342)(0.5-0.423)(0.5-0.65)} \cdot (30) + \frac{(0.39-0.342)(0.39-0.423)(0.39-0.5)}{(0.65-0.342)(0.65-0.423)(0.65-0.5)} \cdot (35)$$

$$\Rightarrow x = 22.84057797$$

Example 3.8.2 Use **Newton's Divided Difference Inverse Interpolation Formula** to find the value of x for $f(x) = 10$ from the given data: $f(0) = 6$, $f(3) = 120$, $f(5) = 336$ & $f(7) = 720$.

Solution: Construct **Newton's Divided Difference (DD) Table**:

	y	$F[y]$	Newton's 1 st DD	Newton's 2 nd DD	Newton's 3 rd DD
y_0	6	0			
			$\frac{3-0}{120-6} = \frac{1}{38}$		
y_1	120	3		$\frac{\frac{1}{108} - \frac{1}{38}}{336-6} = -\frac{7}{135432}$	
			$\frac{5-3}{336-120} = \frac{1}{108}$		$\frac{-\frac{7}{1036800} - \left(-\frac{7}{135432}\right)}{720-6} \approx 0$
y_2	336	5		$\frac{\frac{1}{192} - \frac{1}{108}}{720-120} = -\frac{7}{1036800}$	
			$\frac{7-5}{720-336} = \frac{1}{192}$		
y_3	720	7			

Newton's Inverse Divided Difference Interpolation Formula: $y = f(x) = 10$

$$F(y) = F[y_0] + (y - y_0) \cdot F[y_0, y_1] + (y - y_0)(y - y_1) \cdot F[y_0, y_1, y_2] + (y - y_0)(y - y_1)(y - y_2) \cdot F[y_0, y_1, y_2, y_3]$$

$$\Rightarrow x = 0 + (10 - 6) \left(\frac{1}{38} \right) + (10 - 6)(10 - 120) \left(-\frac{7}{135432} \right) + (10 - 6)(10 - 120)(10 - 336) \left(0 \right)$$

$$\Rightarrow x = \frac{197}{1539} = 0.128$$

Example 3.8.3: Use appropriate **Lagrange's Interpolating Method** to find the time t when distance $x = f(t) = 100$ kms from the following data.

t (in second)	0	1	2	3
$x = f(t)$ (in kms)	0	20	60	120

Solution: We have

	$F(x_0)$	$F(x_1)$	$F(x_2)$	$F(x_3)$
$F(x) = t$	0	1	2	3
$x = f(t)$	0	20	60	120
	x_0	x_1	x_2	x_3

Given $x = f(t) = 100$ and $t = ?$

Using **Lagrange Inverse Interpolating Formula**

$$t = F(x) = L_0(x) \cdot F(x_0) + L_1(x) \cdot F(x_1) + L_2(x) \cdot F(x_2) + L_3(x) \cdot F(x_3)$$

$$\Rightarrow t = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot F(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot F(x_1) \\ + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot F(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot F(x_3)$$

$$\Rightarrow t = \frac{(100-20)(100-60)(100-120)}{(0-20)(0-60)(0-120)} \cdot (0) + \frac{(100-0)(100-60)(100-120)}{(20-0)(20-60)(20-120)} \cdot (1) \\ + \frac{(100-0)(100-20)(100-120)}{(60-0)(60-20)(60-120)} \cdot (2) + \frac{(100-0)(100-20)(100-60)}{(120-0)(120-20)(120-60)} \cdot (3)$$

$$\Rightarrow t = 0 - 1 + \frac{20}{9} + \frac{4}{5}$$

$$\Rightarrow t = \frac{23}{9} = 2.55555555 \text{ Second}$$

Example 3.8.4: Use appropriate **Newton's Divided Difference Interpolating Formula** to find the time t when distance $x = f(t) = 100$ kms from the following data.

t (in second)	0	1	2	3
$x = f(t)$ (in kms)	0	20	60	120

Solution: We have

	$F(x_0)$	$F(x_1)$	$F(x_2)$	$F(x_3)$
$F(x) = t$	0	1	2	3
$x = f(t)$	0	20	60	120
	x_0	x_1	x_2	x_3

Given $x = f(t) = 100$ and $t = ?$

Construct Newton's Divided Difference (DD) Table:

	x	$F[x]$	Newton's 1 st DD	Newton's 2 nd DD	Newton's 3 rd DD
x_0	0	0			
			$\frac{1-0}{20-0} = \frac{1}{20}$		
x_1	20	1		$\frac{\frac{1}{40} - \frac{1}{20}}{60-0} = \frac{-1}{2400}$	
			$\frac{2-1}{60-20} = \frac{1}{40}$		$\frac{\frac{-1}{12000} - \frac{-1}{2400}}{120-0} = \frac{1}{360000}$
x_2	60	2		$\frac{\frac{1}{60} - \frac{1}{40}}{120-20} = \frac{-1}{12000}$	
			$\frac{3-2}{120-60} = \frac{1}{60}$		
x_3	120	3			

From Newton's divided difference (DD) table, we have:

$$F[x_0] = 0, F[x_0, x_1] = \frac{1}{20}, F[x_0, x_1, x_2] = \frac{-1}{2400}, F[x_0, x_1, x_2, x_3] = \frac{1}{360000}$$

Substitute these in the **Newton's divided difference interpolation formula**, we get-

$$t = F(x) = F[x_0] + (x - x_0)F[x_0, x_1] + (x - x_0)(x - x_1)F[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)F[x_0, x_1, x_2, x_3]$$

$$\Rightarrow t = F(100) = 0 + (100 - 0)\left(\frac{1}{20}\right) + (100 - 0)(100 - 20)\left(\frac{-1}{2400}\right) + (100 - 0)(100 - 20)(100 - 60)\left(\frac{1}{360000}\right)$$

$$\Rightarrow t = F(100) = 2\frac{5}{9} = 2.555555556 \text{ Second}$$

Exercise 1: Interpolation

Type-I: Choose the correct answer: (Questions 1-6)

1. The relation between operators: $\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) \equiv$
 - (a) μ
 - (b) 2μ
 - (c) δ
 - (d) None of these
2. The expression: $(1 + \Delta)(1 - \nabla)$ is equal to:
 - (a) 1
 - (b) -1
 - (c) $\Delta\nabla$
 - (d) None of these
3. The relation between operators: $\Delta\nabla \equiv$:
 - (a) $\nabla\Delta$
 - (b) δ^2
 - (c) $4(\mu^2 - 1)$
 - (d) All of these
4. If $f(-2) = 1$ and $f(-1) = 4$, then the **Lagrange's interpolating polynomials** of degree **one** is given by :
 - (b) $7x - 3$
 - (c) $3x + 7$
 - (d) $3x - 7$
 - (e) $7x + 3$
5. If $f(1.5) = -0.25$, $f(3) = 2$ and $f(6) = 20$ then **Newton's Divided Difference Interpolation Formula** will approximate $f(4)$ by:
 - (a) 4
 - (b) 5
 - (c) 6
 - (d) 7
6. If $f(0) = 1$, $f(1) = 2$, $f(2) = 1$ and $f(3) = 10$ then from **Forward Difference Table**, $\Delta^3 f(0)$ is equal to:
 - (a) -21
 - (b) 12
 - (c) -13
 - (d) 31

Type-II: Solve the following: (Questions 7 -16)

7. Prove the following relations between operators with usual notations:

- (a) $E \equiv 1 + \Delta$
- (b) $\Delta \nabla \equiv \nabla \Delta \equiv \delta^2$
- (c) $\delta^2 \equiv 4(\mu^2 - 1)$
- (d) $\mu \delta \equiv \frac{1}{2}(\Delta + \nabla)$
- (e) $\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)(1 + \Delta)^{\frac{1}{2}} \equiv 2 + \Delta$
- (f) $\Delta + \nabla \equiv \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$
- (g) $(1 + \Delta)(1 - \nabla) \equiv 1$

8. For the following data:

- (a) Approximate $f(0.05)$ using Newton forward-difference formula
- (b) Approximate $f(0.65)$ using Newton backward-difference formula

x	0.0	0.2	0.4	0.6	0.8
$f(x)$	1.00000	1.22140	1.49182	1.82212	2.22554

9. Use appropriate Lagrange interpolating polynomials of degree one, two and three to approximate each of the following:

- a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$
- b. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$
- c. $f(0.25)$ if $f(0.1) = 0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$
- d. $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$

10. Use appropriate Lagrange interpolating polynomials of degree one, two and three to approximate each of the following:

- a. $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
- b. $f(0)$ if $f(-0.5) = 1.93750$, $f(-0.25) = 1.33203$, $f(0.25) = 0.800781$, $f(0.5) = 0.687500$
- c. $f(0.18)$ if $f(0.1) = -0.29004986$, $f(0.2) = -0.56079734$, $f(0.3) = -0.81401972$, $f(0.4) = -1.0526302$
- d. $f(0.25)$ if $f(-1) = 0.86199480$, $f(-0.5) = 0.95802009$, $f(0) = 1.0986123$, $f(0.5) = 1.2943767$

11. Show that the polynomial interpolating the following data has degree 3.

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

12. For a function f , the Newton divided-difference formula gives the interpolating polynomial

$$P_3(x) = 1 + 4x + 4x(x - 0.25) + \frac{16}{3}x(x - 0.25)(x - 0.5),$$

on the nodes $x_0 = 0, x_1 = 0.25, x_2 = 0.5$ and $x_3 = 0.75$. Find $f(0.75)$.

13. For a function f , the divided differences are given by

$x_0 = 0.0$	$f[x_0]$		
		$f[x_0, x_1]$	
$x_1 = 0.4$	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{50}{7}$
		$f[x_1, x_2] = 10$	
$x_2 = 0.7$	$f[x_2] = 6$		

Determine the missing entries in the table.

14. A census of the population of the United States is taken in every 10 years. The following table lists the population, in thousands of people, from 1950 to 2000

Year	1950	1960	1970	1980	1990	2000
Population (in thousands)	151,326	179,323	203,302	226,542	249,633	281,422

Use Newton divided-difference formula to approximate the population in the year 1975.

15. Using **Newton's Divided Difference Interpolating Formula**, find the value of x for $f(x) = 0.390$, If $f(20) = 0.342, f(25) = 0.435, f(30) = 0.5$ and $f(35) = 0.65$
16. Use **Newton's Divided Difference Interpolating Formula** to find a polynomial function satisfying the following data: $f(-4) = 1245, f(-1) = 33, f(0) = 5, f(2) = 9$ and $f(5) = 1335$. Hence find $f(-3)$.

Numerical Analysis-1

(For Science College Students)

MATH-316

Chapter-4: Numerical Differentiation and Integration

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Chapter-4: Numerical Differentiation and Integration

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4.1 Direct Formulae for Numerical Differentiation:

(a) Formulae for Derivatives using Forward Difference Operator:

We know that the **Newton's Forward Difference Interpolation** formula is given by-

$$f(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0) + \dots \quad \dots (1)$$

Where, $s = \frac{x-x_0}{h} \quad \dots (2)$

Differentiating eqn ... (1) with respect to s , we get-

$$\frac{d}{ds} f(x) = \Delta f(x_0) + \frac{(2s-1)}{2}\Delta^2 f(x_0) + \frac{(3s^2-6s+2)}{6}\Delta^3 f(x_0) + \dots \quad \dots (3)$$

Differentiating eqn. ... (2) with respect to x , we get-

$$\frac{ds}{dx} = \frac{1}{h} \quad \dots (4)$$

Now we can write

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{ds} [f(x)] \cdot \frac{ds}{dx}$$

$$f'(x) = \frac{1}{h} \left[\Delta f(x_0) + \left(\frac{2s-1}{2} \right) \Delta^2 f(x_0) + \left(\frac{3s^2-6s+2}{6} \right) \Delta^3 f(x_0) + \dots \right] \quad \dots (5)$$

Expression (5) provides the value of $f'(x)$ at any x (which is not tabulated)

For tabulated values of x :

In particular when $x = x_0$ and $s = 0$ Expression (5) provides the value of $f'(x)$ for any tabulated value of x such that-

$$f'(x_0) = \frac{1}{h} \left[\Delta f(x_0) - \frac{1}{2}\Delta^2 f(x_0) + \frac{1}{3}\Delta^3 f(x_0) - \frac{1}{4}\Delta^4 f(x_0) + \frac{1}{5}\Delta^5 f(x_0) - \dots \right] \quad \dots (6)$$

Differentiating eqn. ... (5) with respect to x , we get-

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{ds} [f'(x)] \cdot \frac{ds}{dx}$$

$$= \frac{1}{h} \left[\Delta^2 f(x_0) + (s-1) \Delta^3 f(x_0) + \left(\frac{6s^2-18s+11}{12} \right) \Delta^4 f(x_0) + \dots \right] \frac{1}{h}$$

$$f''(x) = \frac{1}{h^2} \left[\Delta^2 f(x_0) + (s-1) \Delta^3 f(x_0) + \left(\frac{6s^2-18s+11}{12} \right) \Delta^4 f(x_0) + \dots \right] \quad \dots (7)$$

Putting $x = x_0$ and $s = 0$ in eqn ... (7), we get-

$$f''(x_0) = \frac{1}{h^2} \left[\Delta^2 f(x_0) - \Delta^3 f(x_0) + \frac{11}{12} \Delta^4 f(x_0) - \frac{5}{6} \Delta^5 f(x_0) + \frac{13}{36} \Delta^6 f(x_0) + \dots \right] \quad \dots (8)$$

Similarly, we can get-

$$f'''(x_0) = \frac{1}{h^3} \left[\Delta^3 f(x_0) - \frac{3}{2} \Delta^4 f(x_0) + \dots \right] \quad \dots (9)$$

Example 4.1.1: Given: $E \equiv e^{hD}$ then show that :

$$Df(x_0) = \frac{1}{h} \left[\Delta f(x_0) - \frac{1}{2} \Delta^2 f(x_0) + \frac{1}{3} \Delta^3 f(x_0) - \frac{1}{4} \Delta^4 f(x_0) + \frac{1}{5} \Delta^5 f(x_0) - \dots \right]$$

Solution: **Given** $E \equiv e^{hD}$... (1)

We know that $E \equiv 1 + \Delta$... (2)

From equations ... (1) and ... (2), we can write:

$$e^{hD} \equiv \mathbf{1} + \Delta \quad \dots (3)$$

Taking *log* on both side of equation ... (3), we get-

$$\mathbf{log}_e(e^{hD}) \equiv \mathbf{log}_e(1 + \Delta) \qquad [\mathbf{log}_e(e^{hD}) = \ln(e^{hD})]$$

$$\Rightarrow hD.\log_e(e) \equiv \log_e(1 + \Delta)$$

$$\Rightarrow \quad \textcolor{blue}{hD.1} \equiv \textcolor{red}{\log_e(1 + \Delta)} \quad [\because \textcolor{red}{\log_e(e)} = 1]$$

$$\Rightarrow D \equiv \frac{1}{h} [\log_e(1 + \Delta)]$$

We know that, $\log_e(1 + \Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \dots$

Thus, $D \equiv \frac{1}{h} \left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \dots \infty \right]$

$$\Rightarrow Df(x_0) = \frac{1}{h} \left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \dots \right] f(x_0)$$

$$\Rightarrow Df(x_0) = \frac{1}{h} \left[\Delta f(x_0) - \frac{1}{2} \Delta^2 f(x_0) + \frac{1}{3} \Delta^3 f(x_0) - \frac{1}{4} \Delta^4 f(x_0) + \frac{1}{5} \Delta^5 f(x_0) - \dots \right]$$

This is required answer.

Note: We know that $D \equiv \frac{1}{h} \left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \dots \right]$

$$\Rightarrow \quad D^2 \equiv \frac{1}{h^2} \left[\Delta^2 - \Delta^3 + \frac{11}{12} \Delta^4 - \frac{5}{6} \Delta^5 + \frac{13}{36} \Delta^6 + \dots \right]$$

$$\Rightarrow D^2 f(x_0) = \frac{1}{h^2} \left[\Delta^2 f(x_0) - \Delta^3 f(x_0) + \frac{11}{12} \Delta^4 f(x_0) - \frac{5}{6} \Delta^5 f(x_0) + \frac{13}{36} \Delta^6 f(x_0) + \dots \right]$$

Moreover, $D^3 \equiv \frac{1}{h^3} \left[\Delta^3 - \frac{3}{2} \Delta^4 + \dots \right]$

$$\Rightarrow D^3 f(x_0) = \frac{1}{h^3} \left[\Delta^3 f(x_0) - \frac{3}{2} \Delta^4 f(x_0) + \dots \right]$$

Example 4.1.2: Use the forward difference formulas to determine approximations for $f'(0.1)$ and $f'(0.2)$ from the following data:

x	0.1	0.2	0.3	0.4
$f(x)$	0.9975	0.9900	0.9776	0.9604

Solution: Construct Forward Difference Table:

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	0.1	0.9975	-0.0075	-0.0049	0.0001
x_1	0.2	0.9900	-0.0124	-0.0048	
x_2	0.3	0.9776	-0.0172		
x_3	0.4	0.9604			

Case-1: Here- $x_0 = 0.1$ and $h = 0.1$

Writing the forward difference formula for First derivative-

$$f'(x_0) = \frac{1}{h} \left[\Delta f(x_0) - \frac{1}{2} \Delta^2 f(x_0) + \frac{1}{3} \Delta^3 f(x_0) \right]$$

$$f'(0.1) = \frac{1}{0.1} \left[\Delta f(0.1) - \frac{1}{2} \Delta^2 f(0.1) + \frac{1}{3} \Delta^3 f(0.1) \right]$$

From Table $\Delta f(0.1) = -0.0075$, $\Delta^2 f(0.1) = -0.0049$ $\Delta^3 f(0.1) = 0.0001$

Thus,

$$f'(0.1) = \frac{1}{0.1} \left[-0.0075 - \frac{1}{2}(-0.0049) + \frac{1}{3}(0.0001) \right]$$

$$f'(0.1) = -0.050167$$

Case-2: Here, $x_0 = 0.2$ and $h = 0.1$

Writing the formula for First derivative-

$$f'(x_0) = \frac{1}{h} \left[\Delta f(x_0) - \frac{1}{2} \Delta^2 f(x_0) \right]$$

$$f'(0.2) = \frac{1}{0.1} \left[\Delta f(0.2) - \frac{1}{2} \Delta^2 f(0.2) \right]$$

From Table, $\Delta f(0.2) = -0.0124$, $\Delta^2 f(0.2) = -0.0048$

Thus,

$$f'(0.2) = \frac{1}{0.1} \left[-0.0124 - \frac{1}{2} (-0.0048) \right]$$

$$f'(0.2) = -0.1$$

Example 4.1.3: Use the forward difference formulas to approximate $f''(0.1)$ and $f'''(0.1)$ from the following data:

x	0.1	0.2	0.3	0.4
$f(x)$	0.9975	0.9900	0.9776	0.9604

Solution: Construct Forward Difference Table:

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	0.1	0.9975	-0.0075	-0.0049	0.0001
x_1	0.2	0.9900	-0.0124	-0.0048	
x_2	0.3	0.9776	-0.0172		
x_3	0.4	0.9604			

Case-1: Here- $x_0 = 0.1$ and $h = 0.1$

Writing the forward difference formula for second derivative-

$$f''(x_0) = \frac{1}{h^2} [\Delta^2 f(x_0) - \Delta^3 f(x_0)]$$

$$f''(0.1) = \frac{1}{(0.1)^2} [\Delta^2 f(0.1) - \Delta^3 f(0.1)]$$

From Table, $\Delta^2 f(0.1) = -0.0049$ $\Delta^3 f(0.1) = 0.0001$

Thus,

$$f''(0.1) = \frac{1}{(0.1)^2} [-0.0049 - 0.0001]$$

$$f''(0.1) = -0.5$$

Case-2: Here- $x_0 = 0.1$ and $h = 0.1$

Writing the forward difference formula for third derivative-

$$f'''(x_0) = \frac{1}{h^3} [\Delta^3 f(x_0)]$$

$$f'''(0.1) = \frac{1}{h^3} [\Delta^3 f(0.1)]$$

From Table- $\Delta^3 f(0.1) = 0.0001$

Thus,

$$f'''(0.1) = \frac{1}{(0.1)^3} [0.0001]$$

$$f'(0.2) = 0.1$$

Example 4.1.4: Use **Forward Difference Formulae** to approximate $f'(3)$ and $f''(1)$ from the following table.

x	1	3	5	7	9
$f(x)$	85.3	74.5	67.0	60.5	54.3

Solution: Construct Forward Difference Table:

Forward Difference Table						
	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_0	1	85.3	-10.8	3.3	-2.3	1.6
x_1	3	74.5	-7.5	1.0	-0.7	
x_2	5	67.0	-6.5	0.3		
x_3	7	60.5	-6.2			
x_4	9	54.3				

To Find $f'(3)$: Here- $x_1 = 3$ and $h = 2$

Writing the forward difference formula for first derivative-

$$f'(3) = \frac{1}{h} \left[\Delta f(3) - \frac{1}{2} \Delta^2 f(3) + \frac{1}{3} \Delta^3 f(3) \right]$$

$$\Rightarrow f'(3) = \frac{1}{2} \left[-7.5 - \frac{1}{2}(1.0) + \frac{1}{3}(-0.7) \right]$$

$$\Rightarrow f'(3) = -4.116666667$$

To Find $f''(1)$: Here- $x_0 = 1$ and $h = 2$

Writing the forward difference formula for second derivative-

$$f''(1) = \frac{1}{h^2} \left[\Delta^2 f(1) - \Delta^3 f(1) + \frac{11}{12} \Delta^4 f(1) \right]$$

$$\Rightarrow f''(1) = \frac{1}{2^2} \left[3.3 - (-2.3) + \frac{11}{12}(1.6) \right]$$

$$\Rightarrow f''(1) = 1.766666667$$

(b) Formulae for Derivatives using Backward Difference Operator:

We know that the **Newton's Backward Difference Interpolation formula** is given by-

$$f(x) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!}\nabla^2 f(x_n) + \frac{s(s+1)(s+2)}{3!}\nabla^3 f(x_n) + \dots \quad \dots (1)$$

where, $s = \frac{x-x_n}{h} \quad \dots (2)$

Differentiating eqn ... (1) with respect to s , we get-

$$\frac{d}{ds}f(x) = \nabla f(x_n) + \frac{(2s+1)}{2}\nabla^2 f(x_n) + \frac{(3s^2+6s+2)}{6}\nabla^3 f(x_n) + \dots \quad \dots (3)$$

Differentiating eqn ... (2) with respect to x , we get-

$$\frac{ds}{dx} = \frac{1}{h} \quad \dots (4)$$

Now we can write-

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{ds}[f(x)] \cdot \frac{ds}{dx}$$

$$f'(x) = \frac{1}{h} \left[\nabla f(x_n) + \left(\frac{2s+1}{2} \right) \nabla^2 f(x_n) + \left(\frac{3s^2+6s+2}{6} \right) \nabla^3 f(x_n) + \dots \right] \quad \dots (5)$$

Expression (5) provides the value of $f'(x)$ at any x (which is not tabulated)

For tabulated values of x :

In particular when $x = x_n$ and $s = 0$. Expression (5) provides the value of $f'(x)$ for any tabulated value of x such that-

$$f'(x_n) = \frac{1}{h} \left[\nabla f(x_n) + \frac{1}{2}\nabla^2 f(x_n) + \frac{1}{3}\nabla^3 f(x_n) + \frac{1}{4}\nabla^4 f(x_n) + \frac{1}{5}\nabla^5 f(x_n) + \dots \right] \quad \dots (6)$$

Differentiating eqn ... (5) with respect to x , we get-

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{ds}[f'(x)] \cdot \frac{ds}{dx}$$

$$f''(x) = \frac{1}{h} \left[\nabla^2 f(x_n) + (s+1) \nabla^3 f(x_n) + \left(\frac{6s^2+18s+11}{12} \right) \nabla^4 f(x_n) + \dots \right] \frac{1}{h}$$

$$f''(x) = \frac{1}{h^2} \left[\nabla^2 f(x_n) + (s+1) \nabla^3 f(x_n) + \left(\frac{6s^2+18s+11}{12} \right) \nabla^4 f(x_n) + \dots \right] \quad \dots (7)$$

Putting $x = x_n$ and $s = 0$ in eqn ... (7), we get-

$$f''(x_n) = \frac{1}{h^2} \left[\nabla^2 f(x_n) + \nabla^3 f(x_n) + \frac{11}{12} \nabla^4 f(x_n) + \frac{5}{6} \nabla^5 f(x_n) + \frac{13}{36} \nabla^6 f(x_n) + \dots \right] \quad \dots (8)$$

Similarly, we can get-

$$f'''(x_n) = \frac{1}{h^3} \left[\nabla^3 f(x_n) + \frac{3}{2} \nabla^4 f(x_n) + \dots \right] \quad \dots (9)$$

Example 4.1.5: Use the Backward difference formulas to determine approximations for $f'(0.4)$ and $f'(0.3)$ from the following data:

x	0.1	0.2	0.3	0.4
$f(x)$	0.9975	0.9900	0.9776	0.9604

Solution: Construct Backward Difference Table:

	x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
x_0	0.1	0.9975			
x_1	0.2	0.9900	-0.0075		
x_2	0.3	0.9776	-0.0124	-0.0049	
x_3	0.4	0.9604	-0.0172	-0.0048	0.0001

Case-1: Here- $x_n = x_3 = 0.4$ and $h = 0.1$

Writing the backward difference formula for first derivative-

$$f'(x_n) = \frac{1}{h} \left[\nabla f(x_n) + \frac{1}{2} \nabla^2 f(x_n) + \frac{1}{3} \nabla^3 f(x_n) \right]$$

$$f'(0.4) = \frac{1}{h} \left[\nabla f(0.4) + \frac{1}{2} \nabla^2 f(0.4) + \frac{1}{3} \nabla^3 f(0.4) \right]$$

From Table- $\Delta f(0.4) = -0.0172$, $\Delta^2 f(0.4) = -0.0048$ $\Delta^3 f(0.4) = 0.0001$

Thus,
$$f'(0.4) = \frac{1}{0.1} \left[-0.0172 + \frac{1}{2}(-0.0048) + \frac{1}{3}(0.0001) \right]$$

$$f'(0.1) = -0.19566666$$

Case-2: Here- $x_n = x_2 = 0.3$ and $h = 0.1$

Writing the formula for first derivative-

$$f'(x_n) = \frac{1}{h} \left[\nabla f(x_n) + \frac{1}{2} \nabla^2 f(x_n) \right]$$

$$f'(0.3) = \frac{1}{0.1} \left[\nabla f(0.3) + \frac{1}{2} \nabla^2 f(0.3) \right]$$

From Table- $\nabla f(0.3) = -0.0124$, $\nabla^2 f(0.3) = -0.0049$

Thus,
$$f'(0.3) = \frac{1}{0.1} \left[-0.0124 + \frac{1}{2}(-0.0049) \right] = -0.1485$$

4.2 Numerical Differentiation:

(a) Using Lagrange Interpolating Polynomial

Example 4.2.1: Obtain the **cubic Lagrange's interpolating polynomial** from the following data:

x	0	1	2	5
$f(x)$	2	3	12	147

Hence approximate: $f'(1)$, $f''(1)$ and $f'''(1)$

Solution: Do Yourself

(b) Using Newton Divided Difference Interpolating Formula

Example 4.2.2: Use **Newton Divided Difference Interpolating Polynomial** to approximate: $f'(4)$ and $f''(4)$ from the table:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution: Construct **Newton's Divided Difference (DD) Table:**

	x	$f[x]$	Newton's 1 st DD	Newton's 2 nd DD	Newton's 3 rd DD	Newton's 4 th DD
x_0	4	48				
			$\frac{100 - 48}{5 - 4} = 52$			
x_1	5	100		$\frac{97 - 52}{7 - 4} = 15$		
			$\frac{294 - 100}{7 - 5} = 97$		$\frac{21 - 15}{10 - 4} = 1$	
x_2	7	294		$\frac{202 - 97}{10 - 5} = 21$		$\frac{1 - 1}{11 - 4} = 0$
			$\frac{900 - 294}{10 - 7} = 202$		$\frac{27 - 21}{11 - 5} = 1$	
x_3	10	900		$\frac{310 - 202}{11 - 7} = 27$		$\frac{1 - 1}{13 - 5} = 0$
			$\frac{1210 - 900}{11 - 10} = 310$		$\frac{33 - 27}{13 - 7} = 1$	
x_4	11	1210		$\frac{409 - 310}{13 - 10} = 33$		
			$\frac{2028 - 1210}{13 - 11} = 409$			
x_5	13	2028				

From Newton's divided difference (DD) table, we have:

$$f[x_0] = 48, \quad f[x_0, x_1] = 52, \quad f[x_0, x_1, x_2] = 15, \quad f[x_0, x_1, x_2, x_3] = 1 \text{ and} \\ f[x_0, x_1, x_2, x_3, x_4] = 0$$

Substitute in Newton's divided difference interpolation formula:

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\ + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \\ + (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3, x_4]$$

$$\Rightarrow f(x) = 48 + (x - 4)52 + (x - 4)(x - 5)15 + (x - 4)(x - 5)(x - 7)1 + 0 + 0$$

$$\Rightarrow f(x) = 48 + (x - 4)52 + (x^2 - 9x + 20)15 + (x^3 - 16x^2 + 83x - 140)$$

$$\Rightarrow f(x) = x^3 - x^2$$

$$\Rightarrow f'(x) = 3x^2 - 2x$$

$$\Rightarrow f'(4) = 3 \cdot 4^2 - 2 \cdot 4$$

$$\Rightarrow f'(4) = 40$$

Next,

$$\Rightarrow f''(x) = 6x - 2$$

$$\Rightarrow f''(4) = 6 \cdot 4 - 2$$

$$\Rightarrow f''(4) = 22$$

4.3 Exercise 4.1: Numerical Differentiation

Type-I: Choose the correct answer: (Questions 1 -3)

- Using **Newton's forward difference formula for derivative** to find $f'(1.5)$, if $\Delta f(1.5) = 3.625$, $\Delta^2 f(1.5) = 3.0$, $\Delta^3 f(1.5) = 0.75$, $\Delta^4 f(1.5) = 0$, and $h = 0.5$
 - 4.00
 - 4.75
 - 3.75
 - None of these
- Given:** $\Delta f(1.0) = 4.6$, $\Delta^2 f(1.0) = -0.2$, $\Delta^3 f(1.0) = 0.1$, $\Delta^4 f(1.0) = 0.1$ and step-size $h = 0.1$, then the value of $f'(1.0)$ is equal to:
 - 0.470833333
 - 4.708333333
 - 47.08333333
 - 470.83333333
- Using **Newton's forward difference formula for derivative** to find $f''(1.5)$, if $\Delta f(1.5) = 3.625$, $\Delta^2 f(1.5) = 3.0$, $\Delta^3 f(1.5) = 0.75$, $\Delta^4 f(1.5) = 0$, and $h = 0.5$
 - 8.00
 - 6.00
 - 9.00
 - None of these

Type-II: Solve the following: (Questions 4 -8)

- Use the forward difference formulas to approximate-
 - $f'(1.0)$
 - $f'(1.2)$
 - $f''(1.0)$
 - $f'''(1.0)$

From the given data:

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
f(x)	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

5. Use the backward difference formulas to approximate-

- (a) $f'(2.2)$
- (b) $f'(2.0)$
- (c) $f''(2.2)$
- (d) $f'''(2.2)$

From the given data:

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
$f(x)$	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

6. Use appropriate interpolation formula to approximate-

- (a) $f'(0.2)$
- (b) $f''(0)$

From the following table:

x	0.0	0.2	0.4	0.6	0.8	1
$f(x)$	1	1.16	3.56	13.96	41.96	101

7. Use appropriate interpolation formula to approximate-

- (a) $f'(2.0)$
- (b) $f''(2.2)$

From the following table

x	1.4	1.6	1.8	2.0	2.2
$y(x)$	4.0552	4.953	6.0496	7.3891	9.0250

8. Use **Newton's Divided Difference Interpolating Formula** to find a polynomial function satisfying the following data: $f(-4) = 1245$, $f(-1) = 33$, $f(0) = 5$, $f(2) = 9$ and $f(5) = 1335$. Hence approximate the following-

- (a) $f'(-3)$
- (b) $f''(-2)$
- (c) $f'''(1)$

4.4 Numerical Integration:

Consider a function of single variable, $y = f(x)$. If the function is known and simple, we can easily obtain its **derivatives** or can evaluate its **definite integral**. However, if we do not know the function as such or the function is complicated and is given in a tabular form $x_i, f(x_i)$, $i = 0, 1, 2, \dots, n$ then we use only **numerical methods** for differentiation and integration.

4.5 Newton-Cotes Integration Formula (General Quadrature Formula):

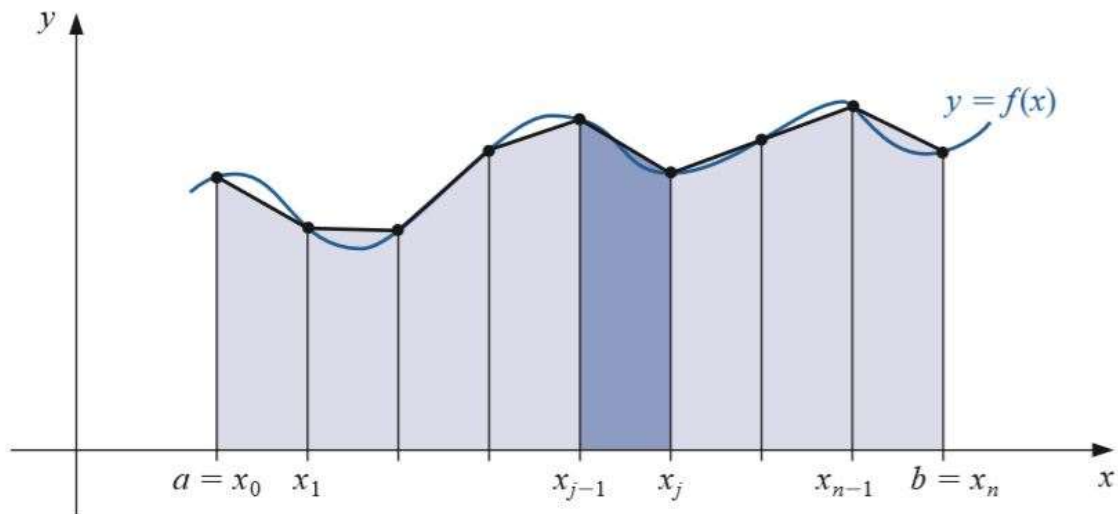
Let us consider a function $y = f(x)$ in the interval $[a, b]$

We divide the interval $[a, b]$ into n equal parts such that:

$$h = \frac{b-a}{n}, \quad x_0 = a, \quad x_n = b, \quad \left[\because h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h} \right]$$

We can write the function $y = f(x)$ as follows:

x	$a = x_0$	x_1	x_2	x_3	$\dots$	$x_n = b$
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\dots$	$f(x_n)$



$[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$ are n subintervals.

Suppose we wish to evaluate the definite integral-

$$I = \int_a^b f(x) dx = \int_{x_0}^{x_0+nh} f(x) dx$$

$$\text{Let } \frac{x-x_0}{h} = s \quad \Rightarrow \quad x = x_0 + sh \quad \Rightarrow \quad dx = h ds$$

When $x = x_0 \Rightarrow s = 0$

When $x = x_0 + nh \Rightarrow s = n$

Thus,

$$I = \int_a^b f(x) dx = \int_{x_0}^{x_0+nh} f(x) dx = h \int_0^n f(x_0 + sh) ds$$

Now, substitute $f(x_0 + sh)$ from **Newton's Forward Difference Interpolation Formula**, we have-

$$\begin{aligned} I &= \int_a^b f(x) dx = \int_{x_0}^{x_0+nh} f(x) dx \\ &= h \int_0^n \left[f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!} \Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!} \Delta^3 f(x_0) + \right. \\ &\quad \left. \frac{s(s-1)(s-2)(s-3)}{4!} \Delta^4 f(x_0) + \dots \right] ds \\ &= h \int_0^n \left[f(x_0) + s\Delta f(x_0) + \frac{(s^2-s)}{2} \Delta^2 f(x_0) + \frac{(s^3-3s^2+2s)}{6} \Delta^3 f(x_0) + \right. \\ &\quad \left. \frac{(s^4-6s^3+11s^2-6s)}{24} \Delta^4 f(x_0) + \dots \right] ds \end{aligned}$$

Integrating with respect to s we get-

$$\begin{aligned} I &= \int_a^b f(x) dx = \int_{x_0}^{x_0+nh} f(x) dx \\ &= h \left[sf(x_0) + \frac{s^2}{2} \Delta f(x_0) + \left(\frac{s^3}{6} - \frac{s^2}{4} \right) \Delta^2 f(x_0) + \left(\frac{s^4}{4} - s^3 + s^2 \right) \frac{1}{6} \Delta^3 f(x_0) + \right. \\ &\quad \left. \left(\frac{s^5}{5} - \frac{3s^4}{2} - \frac{11}{3}s^3 - 3s^2 \right) \frac{1}{24} \Delta^4 f(x_0) + \dots \right]_0^n \end{aligned}$$

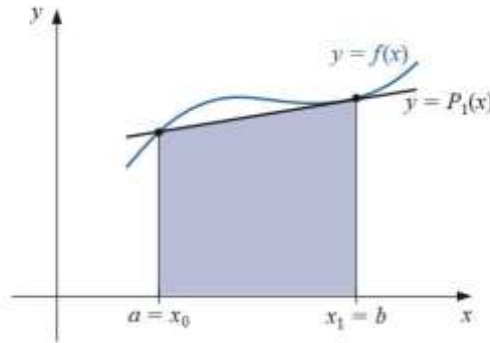
Hence,

$$\begin{aligned} I &= \int_a^b f(x) dx = \int_{x_0}^{x_n} f(x) dx \\ &= h \left[nf(x_0) + \frac{n^2}{2} \Delta f(x_0) + \left(\frac{n^3}{6} - \frac{n^2}{4} \right) \Delta^2 f(x_0) + \left(\frac{n^4}{4} - n^3 + n^2 \right) \frac{1}{6} \Delta^3 f(x_0) \right. \\ &\quad \left. + \left(\frac{n^5}{5} - \frac{3n^4}{2} - \frac{11}{3}n^3 - 3n^2 \right) \frac{1}{24} \Delta^4 f(x_0) + \dots \right] \end{aligned}$$

This is Newton-Cotes General Quadrature Formula.

4.6 Trapezoidal Rule:

Taking $n = 1$ in Newton-Cotes General Quadrature Formula



The curve becomes straight line and thus we get-

$$\begin{aligned}\int_{x_0}^{x_1} f(x) dx &= h \left[f(x_0) + \frac{1}{2} \Delta f(x_0) \right] \\ &= h \left[f(x_0) + \frac{1}{2} f(x_1) - \frac{1}{2} f(x_0) \right] \quad [\because \Delta f(x_0) = f(x_1) - f(x_0)]\end{aligned}$$

$$\int_{x_0}^{x_1} f(x) dx = \frac{h}{2} [f(x_0) + f(x_1)]$$

Similarly,

$$\int_{x_1}^{x_2} f(x) dx = \frac{h}{2} [f(x_1) + f(x_2)]$$

$$\int_{x_2}^{x_3} f(x) dx = \frac{h}{2} [f(x_2) + f(x_3)]$$

$$\begin{matrix} \dots & \dots & \dots & \dots \\ \int_{x_{n-1}}^{x_n} f(x) dx &= & \frac{h}{2} [f(x_{n-1}) + f(x_n)] \end{matrix}$$

Adding up all above integrals-

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \dots + f(x_{n-1}) + f(x_n)]$$

$$\int_a^b f(x) dx = \frac{h}{2} \{f(x_0) + f(x_n) + 2[f(x_1) + f(x_2) + f(x_3) + \dots + f(x_{n-1})]\}$$

This is Trapezoidal Rule.

Example 4.6.1: Approximate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with $h = \frac{1}{4}$

Solution: We can write the function $f(x) = \frac{1}{1+x^2}$ in tabulated form with $h = \frac{1}{4}$

$$\because h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h} = \frac{1-0}{\frac{1}{4}} = 4$$

	x_0	x_1	x_2	x_3	x_4
x	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$f(x) = \frac{1}{1+x^2}$	1	$\frac{16}{17}$	$\frac{4}{5}$	$\frac{16}{25}$	$\frac{1}{2}$
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$

Using Trapezoidal Rule:

$$\int_a^b f(x) dx = \frac{h}{2} [f(x_0) + f(x_n) + 2[f(x_1) + f(x_2) + f(x_3)]]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\frac{1}{4}}{2} \left[1 + \frac{1}{2} + 2 \left[\frac{16}{17} + \frac{4}{5} + \frac{16}{25} \right] \right]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{1}{8} \left[1 + \frac{1}{2} + 2 \left[\frac{1012}{425} \right] \right]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{1}{8} \left[\frac{5323}{850} \right] = \frac{5323}{6800} = 0.78279411$$

Example 4.6.2: Approximate the following integral using **Trapezoidal rule** with $n = 5$

$$\int_0^{\pi} x \cos(x^2) dx$$

Solution: We have: $a = 0, \quad b = \pi, \quad n = 5 \quad \Rightarrow \quad h = \frac{b-a}{n} = \frac{\pi-0}{5} = \frac{\pi}{5}$

Thus, we can write the function $f(x) = x \cos(x^2)$ in tabulated form with $h = \frac{\pi}{5}$

	x_0	x_1	x_2	x_3	x_4	x_5
x	0	$\frac{\pi}{5}$	$\frac{2\pi}{5}$	$\frac{3\pi}{5}$	$\frac{4\pi}{5}$	π
$f(x) = x \cos(x^2)$	0	0.57998801	-0.0104807	-1.72762947	2.51187562	-2.8358697
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$

Using **Trapezoidal Rule:**

$$\int_a^b f(x) dx = \frac{h}{2} [f(x_0) + f(x_5) + 2[f(x_1) + f(x_2) + f(x_3) + f(x_4)]]$$

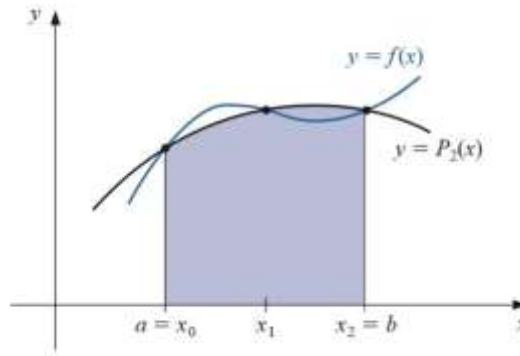
$$\int_0^{\pi} x \cos(x^2) dx = \frac{\pi}{10} [0 - 2.83586970 + 2[0.57998801 - 0.01048070 - 1.72762947 + 2.51187562]]$$

$$\int_0^{\pi} x \cos(x^2) dx = \frac{\pi}{10} [0 - 2.83586970 + 2.70750692]$$

$$\int_0^{\pi} x \cos(x^2) dx = -0.04032635666$$

4.7 Simpson's One-Third Rule:

Taking $n = 2$ in Newton-Cotes General Quadrature Formula



The curve becomes quadratic polynomial and thus we get-

$$\begin{aligned} \int_{x_0}^{x_2} f(x) dx &= h \left[2f(x_0) + \frac{2^2}{2} \Delta f(x_0) + \left(\frac{2^3}{6} - \frac{2^2}{4} \right) \Delta^2 f(x_0) \right] \\ &= h \left[2f(x_0) + 2[f(x_1) - f(x_0)] + \frac{1}{3}[f(x_2) - 2f(x_1) + f(x_0)] \right] \end{aligned}$$

$$\int_{x_0}^{x_2} f(x) dx = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)]$$

Similarly,

$$\int_{x_2}^{x_4} f(x) dx = \frac{h}{3} [f(x_2) + 4f(x_3) + f(x_4)]$$

$$\int_{x_4}^{x_6} f(x) dx = \frac{h}{3} [f(x_4) + 4f(x_5) + f(x_6)]$$

... ..

$$\int_{x_{n-2}}^{x_n} f(x) dx = \frac{h}{3} [f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Adding up all above integrals-

$$\begin{aligned} \int_{x_0}^{x_n} f(x) dx &= \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) \\ &\quad + 4f(x_{n-1}) + f(x_n)] \end{aligned}$$

$$\int_a^b f(x) dx = \frac{h}{3} \{f(x_0) + f(x_n) + 2[f(x_2) + f(x_4) + f(x_6) + \dots + f(x_{n-2})] + 4[f(x_1) + f(x_3) + f(x_5) + \dots + f(x_{n-1})]\}$$

This is Simpson's $\frac{1}{3}$ (One-Third) Rule.

Example 4.7.1: Approximate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's $\frac{1}{3}$ rule with $n = 4$. Hence approximate the value of π

Solution: We have $a = 0$, $b = 1$ so $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

Thus, We can write the function $f(x) = \frac{1}{1+x^2}$ in tabulated form with $h = \frac{1}{4}$

	x_0	x_1	x_2	x_3	x_4
x	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$f(x) = \frac{1}{1+x^2}$	1	$\frac{16}{17}$	$\frac{4}{5}$	$\frac{16}{25}$	$\frac{1}{2}$
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$

Using Simpson's $\frac{1}{3}$ Rule:

$$\int_a^b f(x) dx = \frac{h}{3} [f(x_0) + f(x_n) + 2[f(x_2)] + 4[f(x_1) + f(x_3)]]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{1}{3} \left[1 + \frac{1}{2} + 2 \left[\frac{4}{5} \right] + 4 \left[\frac{16}{17} + \frac{16}{25} \right] \right]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{1}{12} \left[1 + \frac{1}{2} + 2 \left[\frac{4}{5} \right] + 4 \left[\frac{672}{425} \right] \right]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.78539215 \quad \dots (1)$$

Deduction: We know that $\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_{x=0}^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$$\Rightarrow \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} \quad \dots (2)$$

From Equations... (1) & (2), we get $\frac{\pi}{4} = 0.78539215 \Rightarrow \pi = 3.1415686$

Example 4.7.2: Approximate the following integral using **Simpson's $\frac{1}{3}$ Rule** with $n = 6$

$$\int_{0.4}^{1.6} (e^x + \cos x - \log_e x) dx$$

Solution: We have: $a = 0.4$, $b = 1.6$, $n = 6$

$$\Rightarrow h = \frac{b-a}{n} = \frac{1.6-0.4}{6} = 0.2$$

Thus, we can write the function $f(x) = e^x + \cos x - \log_e x$ in tabulated form with $h = 0.2$ as follows:

	x	$f(x) = e^x + \cos x - \log_e x$	
x_0	0.4	3.329176424	$f(x_0)$
x_1	0.6	3.158280039	$f(x_1)$
x_2	0.8	3.145391189	$f(x_2)$
x_3	1.0	3.258584134	$f(x_3)$
x_4	1.2	3.50015312	$f(x_4)$
x_5	1.4	3.888694873	$f(x_5)$
x_6	1.6	4.453829273	$f(x_6)$

Using **Simpson's $\frac{1}{3}$ Rule:**

$$\int_a^b f(x) dx = \frac{h}{3} \{f(x_0) + f(x_6) + 2[f(x_2) + f(x_4)] + 4[f(x_1) + f(x_3) + f(x_5)]\}$$

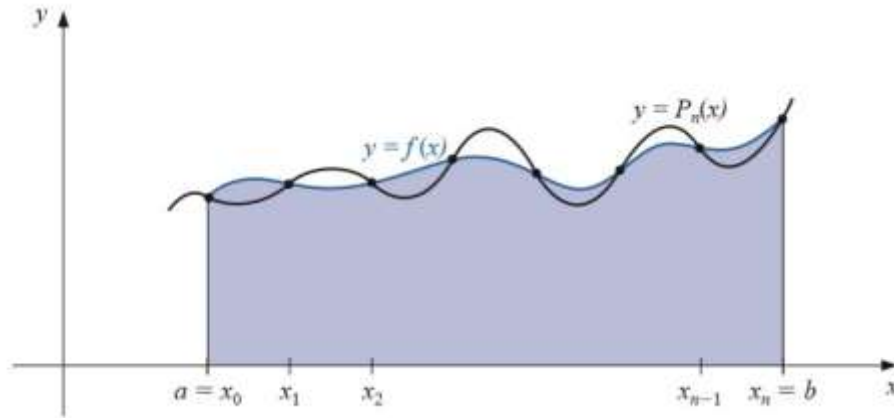
$$\Rightarrow \int_{0.4}^{1.6} (e^x + \cos x - \log_e x) dx = \frac{(0.2)}{3} \{3.329176424 + 4.453829273 + 2[3.145391189 + 3.50015312] + 4[3.158280039 + 3.258584134 + 3.888694873]\}$$

$$\Rightarrow \int_{0.4}^{1.6} (e^x + \cos x - \log_e x) dx = 4.153088793$$

4.8 Simpson's Three-Eighth Rule:

Taking $n = 3$ in Newton-Cotes General Quadrature Formula

Here, the curve becomes a polynomial of **degree 3**. We divide the interval $[a, b]$ into n sub-intervals, where n is divisible by 3.



We consider three strips at a time.

$$\int_{x_0}^{x_3} f(x) dx = h \left[3f(x_0) + \frac{3^2}{2} \Delta f(x_0) + \left(\frac{3^3}{6} - \frac{3^2}{4} \right) \Delta^2 f(x_0) + \left(\frac{3^4}{4} - 3^3 + 3^2 \right) \frac{1}{6} \Delta^3 f(x_0) \right]$$

$$= h \left[3f(x_0) + \frac{9}{2} [f(x_1) - f(x_0)] + \frac{9}{4} [f(x_2) - 2f(x_1) + f(x_0)] + \frac{3}{8} [f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)] \right]$$

$\Rightarrow$

$$\int_{x_0}^{x_3} f(x) dx = h \left[\frac{3}{8} f(x_0) + \frac{9}{8} f(x_1) + \frac{9}{8} f(x_2) + \frac{3}{8} f(x_3) \right]$$

$$= \frac{3h}{8} \{ f(x_0) + 3[f(x_1) + f(x_2)] + f(x_3) \}$$

Similarly,

$$\int_{x_3}^{x_6} f(x) dx = \frac{3h}{8} \{ f(x_3) + 3[f(x_4) + f(x_5)] + f(x_6) \}$$

$$\int_{x_6}^{x_9} f(x) dx = \frac{3h}{8} \{ f(x_6) + 3[f(x_7) + f(x_8)] + f(x_9) \}$$

...

...

...

$$\int_{x_{n-3}}^{x_n} f(x) dx = \frac{3h}{8} \{f(x_{n-3}) + 3[f(x_{n-2}) + f(x_{n-1})] + f(x_n)\}$$

Adding up all above integrals, we get-

$$\int_a^b f(x) dx = \frac{3h}{8} \{f(x_0) + f(x_n) + 2[f(x_3) + f(x_6) + f(x_9) + \dots + f(x_{n-3})] + 3[f(x_1) + f(x_2) + f(x_4) + f(x_5) + \dots + f(x_{n-1})]\}$$

This is Simpson's $\frac{3}{8}$ (Three-Eighth) Rule.

Example 4.8.1: Approximate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's $\frac{3}{8}$ rule with $h = \frac{1}{6}$

Solution: We have $a = 0$, $b = 1$, $h = \frac{1}{6}$

$$\therefore h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h} = \frac{1-0}{\frac{1}{6}} = 6$$

Thus, we can write the function $f(x) = \frac{1}{1+x^2}$ in tabulated form with $h = \frac{1}{6}$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6} = 1$
$f(x) = \frac{1}{1+x^2}$	1	$\frac{36}{37}$	$\frac{36}{40}$	$\frac{36}{45}$	$\frac{36}{52}$	$\frac{36}{61}$	$\frac{1}{2}$
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$	$f(x_6)$

Using Simpson's $\frac{3}{8}$ Rule:

$$\int_a^b f(x) dx = \frac{3h}{8} \{f(x_0) + f(x_n) + 2[f(x_3)] + 3[f(x_1) + f(x_2) + f(x_4) + f(x_5)]\}$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3 \cdot \frac{1}{6}}{8} \left\{ 1 + \frac{1}{2} + 2 \left[\frac{36}{45} \right] + 3 \left[\frac{36}{37} + \frac{36}{40} + \frac{36}{52} + \frac{36}{61} \right] \right\}$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{1}{16} \left\{ 1 + \frac{1}{2} + 2 \left[\frac{36}{45} \right] + 3[3.1554446] \right\} = 0.785395862$$

Example 4.8.2: Approximate the integral $\int_0^2 (e^{2x} \cdot \sin 3x) dx$ using **Simpson's $\frac{3}{8}$ Rule** with $n = 6$

Solution: We have: $a = 0, \quad b = 2, \quad n = 6$

$$\Rightarrow \quad h = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$$

Thus, we can write the function $f(x) = e^{2x} \cdot \sin 3x$ in tabulated form with $h = \frac{1}{3}$

	x	$f(x) = e^{2x} \sin 3x$	
x_0	0	0	$f(x_0)$
x_1	$\frac{1}{3}$	1.6389616	$f(x_1)$
x_2	$\frac{2}{3}$	3.4495724	$f(x_2)$
x_3	$\frac{3}{3}$	1.0427436	$f(x_3)$
x_4	$\frac{4}{3}$	-10.8918380	$f(x_4)$
x_5	$\frac{5}{3}$	-26.880205	$f(x_5)$
x_6	$\frac{6}{3}$	-15.2555692	$f(x_6)$

Using **Simpson's $\frac{3}{8}$ Rule:**

$$\int_a^b f(x) dx = \frac{3h}{8} \{f(x_0) + 2[f(x_3)] + 3[f(x_1) + f(x_2) + f(x_4) + f(x_5)] + f(x_6)\}$$

$$\int_0^2 (e^{2x} \cdot \sin 3x) dx = \frac{3(\frac{1}{3})}{8} \{[0 + 2(1.0427436) + 3(1.6389616 + 3.4495724 - 10.8918380 - 26.880205) - 15.2555692]\}$$

$$\int_0^2 (e^{2x} \cdot \sin 3x) dx = -13.90257613$$

4.9 Exercise 4.2: Numerical Integration

Type-I: Choose the correct answer: (Questions 1 -7)

1. **Trapezoidal rule** is a special case of **Newton-Cotes Integration Formula** for $n =$

- (a) 1
- (b) 3
- (c) 2
- (d) 6

2. **Approximate** the integral $\int_{-3}^3 x^4 dx$, using **Trapezoidal rule** with $n = 6$

- (a) 98
- (b) 115
- (c) 97
- (d) None of these

3. **Simpson's $\frac{1}{3}$ rule** is a special case of **Newton-Cotes Integration Formula** for $n =$

- (a) 1
- (b) 3
- (c) 2
- (d) 6

4. Using **Simpson's $\frac{3}{8}$ rule** and the given table-

x	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$f(x) = \frac{1}{1+x^2}$	1	$\frac{16}{17}$	$\frac{4}{5}$	$\frac{16}{25}$	$\frac{1}{2}$

The **approximate value** of the integral $\int_0^1 \frac{dx}{1+x^2}$ is:

- (a) 6.002647059
- (b) 3.001323539
- (c) 9.003970588
- (d) None of these

5. The **value** of the integral $\int_0^6 f(x) dx$, by **Simpson's 1/3 rule**, using $f(0)$, $f(3)$ and $f(6)$

where, $f(x) = -4 + [25 - (x - 3)^2]^{1/2}$

- (a) 1
- (b) 4
- (c) 6
- (d) None of these

6. Identify the given Rule:

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{3h}{8} \left\{ \begin{array}{l} f(x_0) + 3[f(x_1) + f(x_2) + f(x_4) + f(x_5) + \cdots + f(x_{n-1})] \\ + 2[f(x_3) + f(x_6) + f(x_9) + \cdots + f(x_{n-3})] + f(x_n) \end{array} \right\}$$

- (a) Trapezoidal Rule
- (b) Simpson's $\frac{1}{3}$ Rule
- (c) Newton-Raphson Rule
- (d) None of these

7. Simpson's $\frac{3}{8}$ rule is a special case of **Newton-Cotes Integration Formula** for $n =$

- (a) 1
- (b) 3
- (c) 2
- (d) 6

Type-II: Solve the following: (Questions 8 -15)

- 8.** Approximate the integral $\int_0^8 \frac{1}{1+x} dx$ using **Trapezoidal rule** with $n = 8$
- 9.** Approximate the integral $\int_{-2}^2 x^2 e^x dx$ using **Simpson's $\frac{1}{3}$ rule** with $n = 4$
- 10.** Approximate the integral $\int_0^6 \frac{1}{1+x^2} dx$ using **Simpson's $\frac{3}{8}$ rule** with $n = 6$
- 11.** Approximate the integral $\int_1^3 \frac{x}{x^2+4} dx$ using **Trapezoidal rule** with $n = 8$
- 12.** Approximate the integral $\int_3^5 \frac{1}{\sqrt{x^2-4}} dx$ using **Simpson's $\frac{1}{3}$ rule** with $n = 8$
- 13.** Approximate the integral $\int_0^2 \frac{2}{x^2+4} dx$ using **Simpson's $\frac{3}{8}$ rule** with $n = 6$
- 14.** Approximate the integral $\int_1^2 x \ln x dx$ using **Trapezoidal rule** with $n = 4$
- 15.** Approximate the integral $\int_0^\pi x^2 \cos x dx$ using **Simpson's $\frac{3}{8}$ rule** with $n = 6$

Numerical Analysis-1

(For Science College Students)

MATH-316

Chapter-5: Numerical Solution of Ordinary Differential Equations

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Chapter-5: Numerical Solution of Ordinary Differential Equations

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5.1 Introduction:

Many problems in science and engineering when formulated in mathematical terms lead to ordinary differential equations with appropriate initial and/or boundary conditions.

For example:

- The trajectory of a ballistic missile
- The motion of an artificial satellite in its orbit
- Theories concerning electrical networks
- Bending of beams
- Stability of aircrafts etc.

are modeled by differential equations.

In general, a linear or nonlinear ordinary differential equation can be written as-

$$\frac{d^n y}{dt^n} = f\left(t, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \dots, \frac{d^{n-1} y}{dt^{n-1}}\right)$$

(a) Initial Value Problem:

The first order ordinary differential equation of the form-

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

is called an Initial value problem.

(b) Reduction of a Higher Order ODE to a System of First Order ODE

Any higher order ordinary differential equation can be reduced to a system of first order ordinary differential equations by substitution. i.e.

Consider second order ordinary differential equations-

$$\frac{d^2 y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right), \quad a \leq t \leq b, \quad y(a) = \alpha, \quad y'(a) = \beta \quad \dots (1)$$

$$\text{Let,} \quad p = \frac{dy}{dt} \quad \Rightarrow \quad \frac{dp}{dt} = \frac{d^2 y}{dt^2} \quad \dots (2)$$

Thus, from equations... (1) & (2), we get first order ordinary differential equation-

$$\begin{aligned} \frac{dp}{dt} &= f(t, y, p), & p(a) &= \beta \\ \frac{dy}{dt} &= p, & y(a) &= \alpha \end{aligned} \quad \dots (3)$$

5.2 Euler's Method:

Euler's Method is the most elementary approximation technique for solving initial value problem (IVP). Although this method is not in practice, the simplicity of its derivation can be used to illustrate the techniques involved in the construction of some of the more advanced techniques, without the cumbersome algebra that accompanies these techniques.

Let us consider the well-posed initial value problem (IVP)

$$y'(t) = \frac{dy}{dt} = f(t, y(t)), \quad a = t_0 \leq t \leq t_n = b, \quad y(t_0) = y_0 \quad \dots (1)$$

Define, the common distance h between the points that is call step size-

$$h = \frac{b-a}{n} \quad \Rightarrow \quad h = t_{i+1} - t_i \quad (i = 0, 1, 2, \dots, n-1)$$

Expand the function $y(t_i + h)$ using Taylor's series, we get –

$$y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + \frac{h^3}{3!}y'''(t_i) + \dots + \frac{h^n}{n!}y^{(n)}(t_i) + \dots \quad \dots (2)$$

Since h is small, therefore h^2 and higher terms, being a very small can be deleted. Thus, we get-

$$y(t_i + h) = y(t_i) + hy'(t_i)$$

$$\Rightarrow y(t_i + h) = y(t_i) + hf(t_i, y(t_i)) \quad [\because y'(t) = f(t, y(t))]$$

$$\Rightarrow y_{i+1} = y_i + hf(t_i, y_i) \quad \text{for } i = 0, 1, 2, 3 \dots, n-1$$

$$\Rightarrow y_{i+1}^E = y_i + hf(t_i, y_i)$$

This is called Euler's Method for solving initial value problem (IVP)

Example 5.2.1: Use Euler's Method to approximate the solution of given initial value problem at $t = 2$ with $h = 0.5$

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 < t < 2, \quad y(0) = 0.5$$

Solution: We have, $f(t, y) = y - t^2 + 1$

Euler's First Approximation: Here, $t_0 = 0$, $y_0 = y(t_0) = 0.5$ and $h = 0.5$

$$y_1 = y_0 + hf(t_0, y_0)$$

$$\Rightarrow y_1 = 0.5 + 0.5[y_0 - t_0^2 + 1] \quad [\because f(t_0, y_0) = y_0 - t_0^2 + 1]$$

$$\Rightarrow y_1 = 0.5 + 0.5[0.5 - (0)^2 + 1]$$

$$\Rightarrow y_1 = y(t_1) = y(0.5) = 1.25$$

Euler's Second Approximation: Here, $t_1 = 0.5$, $y_1 = y(t_1) = 1.25$ and $h = 0.5$

$$y_2 = y_1 + hf(t_1, y_1)$$

$$\Rightarrow y_2 = 1.25 + 0.5[y_1 - t_1^2 + 1] \quad [\because f(t_1, y_1) = y_1 - t_1^2 + 1]$$

$$\Rightarrow y_2 = 1.25 + 0.5[1.25 - (0.5)^2 + 1]$$

$$\Rightarrow y_2 = y(t_2) = y(1.0) = 2.25$$

Euler's Third Approximation: Here, $t_2 = 1.0$, $y_2 = y(t_2) = 2.25$ and $h = 0.5$

$$y_3 = y_2 + hf(t_2, y_2)$$

$$\Rightarrow y_3 = 2.25 + 0.5[y_2 - t_2^2 + 1] \quad [\because f(t_2, y_2) = y_2 - t_2^2 + 1]$$

$$\Rightarrow y_3 = 2.25 + 0.5[2.25 - (1.0)^2 + 1]$$

$$\Rightarrow y_3 = y(t_3) = y(1.5) = 3.375$$

Euler's Fourth Approximation: Here, $t_3 = 1.5$, $y_3 = y(t_3) = 3.375$ and $h = 0.5$

$$y_4 = y_3 + hf(t_3, y_3)$$

$$\Rightarrow y_4 = 3.375 + 0.5[y_3 - t_3^2 + 1] \quad [\because f(t_3, y_3) = y_3 - t_3^2 + 1]$$

$$\Rightarrow y_4 = 3.375 + 0.5[3.375 - (1.5)^2 + 1]$$

$$\Rightarrow y_4 = y(t_4) = y(2.0) = 4.4375$$

Example 5.2.2: Apply Euler's Method to approximate $y(0.1)$ to the given initial value problem with $h = 0.02$

$$f(t, y) = \frac{y-t}{y+t}, \quad 0 < t < 0.1, \quad y(0) = 1$$

Solution: We have, $f(t, y) = \frac{y-t}{y+t}$

Euler's First Approximation: Here, $t_0 = 0$, $y_0 = y(t_0) = 1$ and $h = 0.02$

$$y_1 = y_0 + hf(t_0, y_0)$$

$$\Rightarrow y_1 = y_0 + h \left[\frac{y_0 - t_0}{y_0 + t_0} \right] \quad [\because f(t_0, y_0) = \frac{y_0 - t_0}{y_0 + t_0}]$$

$$\Rightarrow y_1 = y(t_1) = y(0.02) = 1.02$$

Euler's Second Approximation: Here, $t_1 = 0.02$, $y_1 = y(t_1) = 1.02$ and $h = 0.02$

$$y_2 = y_1 + hf(t_1, y_1)$$

$$\Rightarrow y_2 = y_1 + h \left[\frac{y_1 - t_1}{y_1 + t_1} \right] \quad [\because f(t_1, y_1) = \frac{y_1 - t_1}{y_1 + t_1}]$$

$$\Rightarrow y_2 = y(t_2) = y(0.04) = 1.0392$$

Euler's Third Approximation: Here, $t_2 = 0.04$, $y_2 = y(t_2) = 1.0392$ and $h = 0.02$

$$y_3 = y_2 + hf(t_2, y_2)$$

$$\Rightarrow y_3 = y_2 + h \left[\frac{y_2 - t_2}{y_2 + t_2} \right] \quad [\because f(t_2, y_2) = \frac{y_2 - t_2}{y_2 + t_2}]$$

$$\Rightarrow y_3 = y(t_3) = y(0.06) = 1.0577$$

Euler's Fourth Approximation: Here, $t_3 = 0.06$, $y_3 = y(t_3) = 1.0577$ and $h = 0.02$

$$y_4 = y_3 + hf(t_3, y_3)$$

$$\Rightarrow y_4 = y_3 + h \left[\frac{y_3 - t_3}{y_3 + t_3} \right] \quad [\because f(t_3, y_3) = \frac{y_3 - t_3}{y_3 + t_3}]$$

$$\Rightarrow y_4 = y(t_4) = y(0.08) = 1.0756$$

Euler's Fifth Approximation: Here, $t_4 = 0.08$, $y_4 = y(t_4) = 1.0756$ and $h = 0.02$

$$y_5 = y_4 + hf(t_4, y_4)$$

$$\Rightarrow y_5 = y_4 + h \left[\frac{y_4 - t_4}{y_4 + t_4} \right] \quad [\because f(t_4, y_4) = \frac{y_4 - t_4}{y_4 + t_4}]$$

$$\Rightarrow y_5 = y(t_5) = y(0.1) = 1.0928$$

5.3 Modified Euler's Method:

The modified Euler's method gives the greater improvement in accuracy over the original Euler's method.

Consider the initial value problem (IVP)-

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(t_0) = y_0 \quad \dots (1)$$

Define, the common distance h between the points that is call step size-

$$h = \frac{b-a}{n} \quad \Rightarrow \quad h = t_{i+1} - t_i \quad (i = 0, 1, 2, \dots, n-1)$$

We know that the Euler's Method is -

$$y_{i+1} = y(t_{i+1}) = y(t_i) + hf(t_i, y_i) \quad \text{For, } i = 0, 1, 2, 3 \dots n-1$$

Then, the Modified Euler's Method is given by-

$$y_{i+1}^{ME} = y(t_{i+1}) = y(t_i) + \frac{h}{2} [f(t_i, y_i) + f(t_{i+1}, y(t_i) + hf(t_i, y_i))]$$

$$\text{For, } i = 0, 1, 2, 3 \dots n-1$$

The Modified Euler's Method can also be expressed as follows-

$$y_{i+1}^{ME} = y(t_{i+1}) = y(t_i) + \frac{h}{2} [f(t_i, y_i) + f(t_{i+1}, y_{i+1}^E)]$$

$$\text{Where, } y_{i+1}^E = y(t_i) + hf(t_i, y_i), \quad \text{For, } i = 0, 1, 2, 3 \dots n-1$$

Example 5.3.1: Use Modified Euler's Method to find $y(0.2)$ and $y(0.4)$ to the given initial value problem, taking $n = 10$

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 < t < 2, \quad y(0) = 0.5$$

Solution: We have, $f(t, y) = y - t^2 + 1$,

Step-1: Here, $t_0 = 0$, $y_0 = y(t_0) = 0.5$ and $h = \frac{b-a}{n} = \frac{2-0}{10} = 0.2$

The Modified Euler's Method for $i = 0$,

$$\begin{aligned} \text{First,} \quad y_1^E &= y(t_0) + hf(t_0, y_0) \\ \Rightarrow y_1^E &= 0.5 + 0.2[y_0 - t_0^2 + 1] & [\because f(t_0, y_0) = y_0 - t_0^2 + 1] \\ \Rightarrow y_1^E &= 0.5 + 0.2[0.5 - (0)^2 + 1] \\ \Rightarrow y_1^E &= 0.8 \end{aligned}$$

Now,

$$\begin{aligned} y_1^{ME} &= y(t_1) = y(t_0 + h) = y(t_0) + \frac{h}{2}[f(t_0, y_0) + f(t_1, y_1^E)] \\ \Rightarrow y_1 &= y(t_1) = y(t_0 + h) = y(t_0) + \frac{h}{2}[(y_0 - t_0^2 + 1) + (y_1^E - t_1^2 + 1)] \\ \Rightarrow y_1 &= y(0.2) = y(0 + 0.2) = 0.5 + \frac{0.2}{2}[(0.5 - (0)^2 + 1) + (0.8 - (0.2)^2 + 1)] \\ \Rightarrow y_1 &= y(0.2) = 0.826 \end{aligned}$$

Step-2: Here, $t_1 = 0.2$, $y_1 = y(0.2) = 0.826$ and $h = 0.2$

The Modified Euler's Method for $i = 1$,

$$\begin{aligned} \text{First,} \quad y_2^E &= y(t_1) + hf(t_1, y_1) \\ \Rightarrow y_2^E &= 0.826 + 0.2[y_1 - t_1^2 + 1] & [\because f(t_1, y_1) = y_1 - t_1^2 + 1] \\ \Rightarrow y_2^E &= 0.826 + 0.2[0.826 - (0.2)^2 + 1] \\ \Rightarrow y_2^E &= 1.1832 \end{aligned}$$

Now,

$$\begin{aligned} y_2^{ME} &= y(t_2) = y(t_1 + h) = y(t_1) + \frac{h}{2}[f(t_1, y_1) + f(t_2, y_2^E)] \\ \Rightarrow y_2 &= y(t_2) = y(t_1 + h) = y(t_1) + \frac{h}{2}[(y_1 - t_1^2 + 1) + (y_2^E - t_2^2 + 1)] \\ \Rightarrow y_2 &= y(0.4) = y(0.2 + 0.2) = 0.826 + \frac{0.2}{2}[(0.826 - (0.2)^2 + 1) + (1.1832 - (0.4)^2 + 1)] \\ \Rightarrow y_2 &= y(0.4) = 1.20692 \end{aligned}$$

Example 5.3.2 Use **Modified Euler's Method** to approximate **y(0.2)** to the given initial value problem:

$$\frac{dy}{dt} = t^2 + y, \quad 0 < t < 1, \quad y(0) = 1, \quad h = 0.1$$

Solution: We have, $f(t, y) = t^2 + y$,

STEP-1: Here, $t_0 = 0$, $t_1 = 0.1$, $y_0 = y(t_0) = 1$ and $h = 0.1$

First from Euler's Formula:

$$y_1^E = y_0 + hf(t_0, y_0)$$

$$\Rightarrow y_1^E = y_0 + h[t_0^2 + y_0] \quad [\because f(t_0, y_0) = t_0^2 + y_0]$$

$$\Rightarrow y_1^E = 1.1$$

Now from Modified Euler's Formula:

$$y_1 = y(t_1) = y_0 + \frac{h}{2}[f(t_0, y_0) + f(t_1, y_1^E)]$$

$$\Rightarrow y_1 = y(t_1) = y_0 + \frac{h}{2}[(t_0^2 + y_0) + (t_1^2 + y_1^E)]$$

$$\Rightarrow y_1 = y(0.1) = 1.1055$$

STEP-2: Here, $t_1 = 0.1$, $t_2 = 0.2$, $y_1 = y(0.1) = 1.1055$ and $h = 0.1$

First from Euler's Formula:

$$y_2^E = y_1 + hf(t_1, y_1)$$

$$\Rightarrow y_2^E = y_1 + h[t_1^2 + y_1] \quad [\because f(t_1, y_1) = t_1^2 + y_1]$$

$$\Rightarrow y_2^E = 1.21705$$

Now from Modified Euler's Formula:

$$y_2 = y(t_2) = y_1 + \frac{h}{2}[f(t_1, y_1) + f(t_2, y_2^E)]$$

$$\Rightarrow y_2 = y(t_2) = y_1 + \frac{h}{2}[(t_1^2 + y_1) + (t_2^2 + y_2^E)]$$

$$\Rightarrow y_2 = y(0.2) = 1.2241275$$

5.4 Taylor's Series Method of Order n:

Consider the first order ordinary differential equation of the form-

$$y'(t) = \frac{dy}{dt} = f(t, y), \quad t_0 \leq t \leq t_n, \quad y(t_0) = y_0 \quad \dots (1)$$

Define, the common distance h between the points that is call step size-

$$h = \frac{b-a}{n} \quad \Rightarrow \quad h = t_{i+1} - t_i \quad (i = 0, 1, 2, \dots, n-1)$$

Expand the function $y(t)$ using Taylor's Polynomial about $t = t_i$, we get –

$$y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + \frac{h^3}{3!}y'''(t_i) + \dots + \frac{h^n}{n!}y^{(n)}(t_i) \quad \dots (2)$$

$\Rightarrow$

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + \frac{h^3}{3!}y'''(t_i) + \dots + \frac{h^n}{n!}y^{(n)}(t_i) \quad \dots (3)$$

Substitute, $y(t_i)$, $y'(t_i)$, $y''(t_i)$... etc from equation ... (1)

The equation ... (3) is called **Taylor's Series Method of Order n** to solve Initial Value Problem (IVP).

Example 5.4.1: Use **Taylor's Series Method of Order 4** to find **$y(0.4)$** to the given initial value problem (**Taking $n = 10$**).

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 < t < 2, \quad y(0) = 0.5$$

Solution: We know that $h = \frac{b-a}{n} = \frac{2-0}{10} = 0.2$

Step-1: Here, $t_0 = 0$ and $y(0) = 0.5$

$$\begin{aligned} y' &= y - t^2 + 1 & \Rightarrow & \quad y'_0 = y_0 - t_0^2 + 1 = 0.5 - 0 + 1 = 1.5 \\ y'' &= y' - 2t & \Rightarrow & \quad y''_0 = y'_0 - 2(t_0) = 1.5 - 2(0) = 1.5 \\ y''' &= y'' - 2 & \Rightarrow & \quad y'''_0 = y''_0 - 2 = 1.5 - 2 = -0.5 \\ y^{(4)} &= y''' & \Rightarrow & \quad y^{(4)}_0 = y'''_0 = -0.5 \end{aligned}$$

Using Taylor's Series Method for $i = 0$

$$\begin{aligned} y_1 &= y(0.2) = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{(4)}_0 \\ \Rightarrow \quad y_1 &= y(0.2) = 0.5 + (0.2)(1.5) + \frac{(0.2)^2}{2!}(1.5) + \frac{(0.2)^3}{3!}(-0.5) + \frac{(0.2)^4}{4!}(-0.5) \\ \Rightarrow \quad y_1 &= y(0.2) = 0.8293 \end{aligned}$$

Step-2: Here, $t_1 = 0.2$ and $y_1 = y(0.2) = 0.8293$

$$\begin{aligned} y'_1 &= y_1 - t_1^2 + 1 = 0.8293 - (0.2)^2 + 1 = 1.7896 \\ y''_1 &= y'_1 - 2(t_1) = 1.7896 - 2(0.2) = 1.3896 \\ y'''_1 &= y''_1 - 2 = 1.3896 - 2 = -0.6104 \\ y^{(4)}_1 &= y'''_1 = -0.6104 \end{aligned}$$

Using Taylor's Series Method for $i = 1$

$$\begin{aligned} y_2 &= y(0.4) = y_1 + hy'_1 + \frac{h^2}{2!}y''_1 + \frac{h^3}{3!}y'''_1 + \frac{h^4}{4!}y^{(4)}_1 \\ \Rightarrow \quad y_2 &= y(0.4) = 0.8293 + (0.2)(1.7896) + \frac{(0.2)^2}{2!}(1.3896) + \frac{(0.2)^3}{3!}(-0.6104) + \frac{(0.2)^4}{4!}(-0.6104) \\ \Rightarrow \quad y_2 &= y(0.4) = 1.214157 \end{aligned}$$

Example 5.4.2: Use **Taylor's Series Method of order 4** to find **$y(0.1)$** to the given initial value problem correct to four decimal places (**Taking $n = 10$**).

$$\frac{dy}{dt} = t - y^2, \quad 0 < t < 1, \quad y(0) = 1$$

Solution: We know that $h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$

Step-1: Here, $t_0 = 0$ and $y_0 = y(0) = 1$

$$y' = t - y^2 \Rightarrow y'_0 = t_0 - y_0^2 = 0 - 1^2 = -1$$

$$y'' = 1 - 2yy' \Rightarrow y''_0 = 1 - 2y_0y'_0 = 1 - 2(1)(-1) = 3$$

$$y''' = -2yy'' - 2(y')^2 \Rightarrow y'''_0 = -2y_0y''_0 - 2(y'_0)^2 = -2(1)3 - 2(-1)^2 = -8$$

$$y^{(4)} = -2yy''' - 6y'y'' \Rightarrow y^{(4)}_0 = -2y_0y'''_0 - 6y'_0y''_0 = -2(1)(-8) - 6(-1)(3) = 34$$

Using Taylor's Series Method for $i = 0$

$$y_1 = y(0.1) = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{(4)}_0$$

$$\Rightarrow y_1 = y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(34)$$

$$\Rightarrow y_1 = y(0.1) = 0.913808333$$

5.5 Runge-Kutta Methods:

The Taylor's Series Methods have the desirable property of high order local truncation error, but the disadvantage of requiring the computation and evaluation of the derivatives of $f(t, y)$. This is a complicated and time consuming procedure for most problems, so Taylor's Series Methods are seldom used in practice.

Computationally, most efficient methods in terms of accuracy were developed by two German Mathematicians, **Carl Runge (1856-1927)** and **Wilhelm Kutta (1867-1944)**. These methods are well known Runge-Kutta Methods.

Runge-Kutta Methods have the high order local truncation error of the Taylors Methods but eliminate the need to compute and evaluate the derivatives of $f(t, y)$.

5.6 Runge-Kutta Second Order Method:

Consider the initial value problem (IVP)-

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(t_0) = y_0 \quad \dots (1)$$

Define, the common distance h between the points that is call step size-

$$h = \frac{b-a}{n} \quad \Rightarrow \quad h = t_{i+1} - t_i \quad (i = 0, 1, 2, \dots, n-1)$$

Second Order Runge-Kutta Method (General Form):

Now, let

$$K_1 = hf(t_i, y_i)$$

and

$$K_2 = h \left[f(t_i + p_1 h, y(t_i) + q_1 hf(t_i, y_i)) \right]$$

$\Rightarrow$

$$K_2 = h[f(t_i + p_1 h, y_i + q_1 K_1)]$$

Hence,

$$y_{i+1} = y_i + (a_1 K_1 + a_2 K_2)$$

Heun's Method:

Taking $p_1 = q_1 = 1$ and $a_1 = a_2 = \frac{1}{2}$ in Second Order Runge-Kutta Method

We get-

$$y_{i+1} = y_i + \left(\frac{K_1 + K_2}{2} \right)$$

Where,

$$K_1 = hf(t_i, y_i)$$

$$K_2 = h[f(t_i + h, y_i + K_1)]$$

Example 5.6.1: Use Heun's Method to find $y(0.2)$ and $y(0.4)$ to the given initial value problem, with $h = 0.2$

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 < t < 2, \quad y(0) = 0.5$$

Solution: We have- $f(t, y) = y - t^2 + 1$,

To find $y(0.2)$ Using Heun's Method: (for $i = 0$)

Here- $t_0 = 0$, $y_0 = y(t_0) = 0.5$ and $h = 0.2$

$$\begin{aligned} \text{First,} \quad & K_1 = hf(t_0, y_0) \\ \Rightarrow & K_1 = h[y_0 - t_0^2 + 1] & [\because f(t, y) = y - t^2 + 1] \\ \Rightarrow & K_1 = 0.2[0.5 - (0)^2 + 1] \\ \Rightarrow & K_1 = 0.3 \end{aligned}$$

$$\begin{aligned} \text{Next,} \quad & K_2 = h[f(t_0 + h, y_0 + K_1)] \\ \Rightarrow & K_2 = h[(y_0 + K_1) - (t_0 + h)^2 + 1] & [\because f(t, y) = (y) - (t)^2 + 1] \\ \Rightarrow & K_2 = 0.2[(0.5 + 0.3) - (0 + 0.2)^2 + 1] \\ \Rightarrow & K_2 = 0.352 \end{aligned}$$

$$\begin{aligned} \text{Now,} \quad & y_1 = y(t_0 + h) = y_0 + \left(\frac{K_1 + K_2}{2}\right) \\ \Rightarrow & y_1 = y(0.2) = 0.5 + \left(\frac{0.3 + 0.352}{2}\right) = 0.826 \end{aligned}$$

To find $y(0.4)$ Using Heun's Method: (for $i = 1$)

Here- $t_1 = 0.2$, $y_1 = y(t_1) = 0.826$ and $h = 0.2$

$$\begin{aligned} \text{First,} \quad & K_1 = hf(t_1, y_1) \\ \Rightarrow & K_1 = h[y_1 - t_1^2 + 1] & [\because f(t, y) = y - t^2 + 1] \\ \Rightarrow & K_1 = 0.2[0.826 - (0.2)^2 + 1] \\ \Rightarrow & K_1 = 0.3572 \end{aligned}$$

$$\begin{aligned} \text{Next,} \quad & K_2 = h[f(t_1 + h, y_1 + K_1)] \\ \Rightarrow & K_2 = h[(y_1 + K_1) - (t_1 + h)^2 + 1] & [\because f(t, y) = (y) - (t)^2 + 1] \\ \Rightarrow & K_2 = 0.2[(0.826 + 0.3572) - (0.2 + 0.2)^2 + 1] \\ \Rightarrow & K_2 = 0.40464 \end{aligned}$$

$$\begin{aligned} \text{Now,} \quad & y_2 = y(t_1 + h) = y_1 + \left(\frac{K_1 + K_2}{2}\right) \\ \Rightarrow & y_2 = y(0.4) = 0.826 + \left(\frac{0.3572 + 0.40464}{2}\right) = 1.20692 \end{aligned}$$

5.7 Runge-Kutta Third Order Method:

Let us consider the initial value problem (IVP)-

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(t_0) = y_0 \quad \dots (1)$$

Define, the common distance h between the points that is call step size-

$$h = \frac{b-a}{n} \quad \Rightarrow \quad h = t_{i+1} - t_i \quad (i = 0, 1, 2, \dots, n-1)$$

The Third order Runge-Kutta Method is described in the following steps:

$$K_1 = hf(t_i, y_i)$$

$$K_2 = hf\left(t_i + \frac{h}{2}, y_i + \frac{K_1}{2}\right)$$

$$K_3 = hf(t_i + h, y_i + K_2'), \quad \text{where} \quad K_2' = hf(t_i + h, y_i + K_1)$$

Hence,

$$y_{i+1} = y_i + \frac{1}{6} [K_1 + 4K_2 + K_3]$$

$$i = 0, 1, 2, \dots, n-1$$

Example 5.7.1: Use **Runge-Kutta Third Order Method** to approximate the solution $y(1.5)$ to the given initial value problem and compare the result to the actual values.

$$y' = \frac{1+t}{1+y}, \quad 1 \leq t \leq 2, \quad y(1) = 2, \quad \text{with } h = 0.5$$

Note: Actual solution $y(t) = \sqrt{(t^2 + 2t + 6)} - 1$

Solution: We have- $f(t, y) = \frac{1+t}{1+y}$

Step-1: Applying **Runge-Kutta Third Order Method** (For $i = 0$)

Here- $t_0 = 1, \quad y_0 = y(t_0) = y(1) = 2 \quad \text{and} \quad h = 0.5$

$$K_1 = hf(t_0, y_0) = h \left[\frac{1+t_0}{1+y_0} \right] = 0.5 \left[\frac{1+1}{1+2} \right] = 0.3333333333$$

$$K_2 = h \left[f \left(t_0 + \frac{h}{2}, y_0 + \frac{K_1}{2} \right) \right] = h \left[\frac{1 + \left(t_0 + \frac{h}{2} \right)}{1 + \left(y_0 + \frac{K_1}{2} \right)} \right] = 0.5 \left[\frac{1 + \left(1 + \frac{0.5}{2} \right)}{1 + \left(2 + \frac{0.3333333333}{2} \right)} \right] = 0.355263158$$

$$K_2' = hf(t_0 + h, y_0 + K_1) = h \left[\frac{1 + (t_0 + h)}{1 + (y_0 + K_1)} \right] = 0.5 \left[\frac{1 + (1 + 0.5)}{1 + (2 + 0.3333333333)} \right] = 0.37500000$$

$$K_3 = hf(t_0 + h, y_0 + K_2') = h \left[\frac{1 + (t_0 + h)}{1 + (y_0 + K_2')} \right] = 0.5 \left[\frac{1 + (1 + 0.5)}{1 + (2 + 0.37500000)} \right] = 0.370370370$$

Thus,

$$\begin{aligned} y_1 &= y_0 + \frac{1}{6} [K_1 + 4(K_2) + K_3] \\ &= 2 + \frac{1}{6} [0.3333333333 + 4(0.355263158) + 0.370370370] \end{aligned}$$

$$y_1 = y(t_0 + h) = y(1.5) = 2.354126055$$

Comparison with the actual values:

Comparison	
Runge-Kutta Third Order Method	Actual values
$y(1.5) = 2.354126055$	$y(1.5) = 2.354101966$

5.8 Runge-Kutta Fourth Order Method:

Let us consider the initial value problem (IVP)-

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(t_0) = y_0 \quad \dots (1)$$

Define, the common distance h between the points that is call step size-

$$h = \frac{b-a}{n} \quad \Rightarrow \quad h = t_{i+1} - t_i \quad (i = 0, 1, 2, \dots, n-1)$$

The Fourth order Runge-Kutta Method is described in the following steps:

$$K_1 = hf(t_i, y_i)$$

$$K_2 = hf\left(t_i + \frac{h}{2}, y_i + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(t_i + \frac{h}{2}, y_i + \frac{K_2}{2}\right)$$

$$K_4 = hf(t_i + h, y_i + K_3)$$

Hence,

$$y_{i+1} = y_i + \frac{1}{6}[K_1 + 2(K_2 + K_3) + K_4]$$

$$i = 0, 1, 2, \dots, n-1$$

Example 5.8.1: Use Runge-Kutta Fourth Order Method to find $y(0.2)$ and $y(0.4)$ to the given initial value problem with $h = 0.2$

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 < t < 2, \quad y(0) = 0.5$$

Solution: We have- $f(t, y) = y - t^2 + 1$,

To find $y(0.2)$ Using Runge-Kutta Fourth Order Method : (Put $i = 0$)

$$\text{Here- } t_0 = 0, \quad y_0 = y(t_0) = 0.5 \quad \text{and} \quad h = 0.2$$

$$K_1 = hf(t_0, y_0) = h[y_0 - t_0^2 + 1] = 0.2[0.5 - (0)^2 + 1]$$

$$K_1 = 0.3$$

$$K_2 = h\left[f\left(t_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)\right] = h\left[\left(y_0 + \frac{K_1}{2}\right) - \left(t_0 + \frac{h}{2}\right)^2 + 1\right] = 0.2\left[\left(0.5 + \frac{0.3}{2}\right) - \left(0 + \frac{0.2}{2}\right)^2 + 1\right]$$

$$K_2 = 0.328$$

$$K_3 = h \left[f \left(t_0 + \frac{h}{2}, y_0 + \frac{K_2}{2} \right) \right] = h \left[\left(y_0 + \frac{K_2}{2} \right) - \left(t_0 + \frac{h}{2} \right)^2 + 1 \right] = 0.2 \left[\left(0.5 + \frac{0.328}{2} \right) - \left(0 + \frac{0.2}{2} \right)^2 + 1 \right]$$

$$K_3 = 0.3308$$

$$K_4 = hf(t_0 + h, y_0 + K_3) = h[(y_0 + K_3) - (t_0 + h)^2 + 1] = 0.2[(0.5 + 0.3308) - (0 + 0.2)^2 + 1]$$

$$K_4 = 0.35816$$

$$\text{Thus, } y_1 = y_0 + \frac{1}{6}[K_1 + 2(K_2 + K_3) + K_4] = 0.5 + \frac{1}{6}[0.3 + 2(0.328 + 0.3308) + 0.35816]$$

$$y_1 = y(t_0 + h) = y(0.2) = 0.8292933$$

To find $y(0.4)$ Using Runge-Kutta Fourth Order Method : (Put $i = 1$)

$$\text{Here- } t_1 = 0.2, \quad y_1 = y(t_1) = 0.8292933 \quad \text{and} \quad h = 0.2$$

$$K_1 = hf(t_1, y_1) = h[y_1 - t_1^2 + 1] = 0.2[0.8292933 - (0.2)^2 + 1] \\ = 0.35785866$$

$$K_2 = h \left[f \left(t_1 + \frac{h}{2}, y_1 + \frac{K_1}{2} \right) \right] = h \left[\left(y_1 + \frac{K_1}{2} \right) - \left(t_1 + \frac{h}{2} \right)^2 + 1 \right] \\ = 0.2 \left[\left(0.8292933 + \frac{0.35785866}{2} \right) - \left(0.2 + \frac{0.2}{2} \right)^2 + 1 \right] = 0.383644526$$

$$K_3 = h \left[f \left(t_1 + \frac{h}{2}, y_1 + \frac{K_2}{2} \right) \right] = h \left[\left(y_1 + \frac{K_2}{2} \right) - \left(t_1 + \frac{h}{2} \right)^2 + 1 \right] \\ = 0.2 \left[\left(0.8292933 + \frac{0.383644526}{2} \right) - \left(0.2 + \frac{0.2}{2} \right)^2 + 1 \right] = 0.3862231126$$

$$K_4 = hf(t_1 + h, y_1 + K_3) = h[(y_1 + K_3) - (t_1 + h)^2 + 1] \\ = 0.2[(0.8292933 + 0.3862231126) - (0.2 + 0.2)^2 + 1] = 0.4111032825$$

Thus,

$$y_2 = y_1 + \frac{1}{6}[K_1 + 2(K_2 + K_3) + K_4]$$

$$y_2 = y(t_1 + h) = y(0.4)$$

$$y_2 = 0.8292933 + \frac{1}{6}[0.35785866 + 2(0.383644526 + 0.3862231126) + 0.4111032825]$$

$$y_2 = y(0.4) = 1.21407617$$

Example 5.8.2: Use **Runge-Kutta Fourth Order Method** to approximate the solution $y(1.5)$ to the given initial value problem and compare the result to the actual values.

$$y' = \frac{1+t}{1+y}, \quad 1 \leq t \leq 2, \quad y(1) = 2, \quad \text{with } h = 0.5$$

Note: Actual solution $y(t) = \sqrt{(t^2 + 2t + 6)} - 1$

Solution: We have- $f(t, y) = \frac{1+t}{1+y}$

Step-1: Applying **Runge-Kutta Fourth Order Method** (For $i = 0$)

Here- $t_0 = 1, \quad y_0 = y(t_0) = y(1) = 2 \quad \text{and} \quad h = 0.5$

$$K_1 = hf(t_0, y_0) = h \left[\frac{1+t_0}{1+y_0} \right] = 0.5 \left[\frac{1+1}{1+2} \right] = 0.3333333333$$

$$K_2 = h \left[f \left(t_0 + \frac{h}{2}, y_0 + \frac{K_1}{2} \right) \right] = h \left[\frac{1 + \left(t_0 + \frac{h}{2} \right)}{1 + \left(y_0 + \frac{K_1}{2} \right)} \right] = 0.5 \left[\frac{1 + \left(1 + \frac{0.5}{2} \right)}{1 + \left(2 + \frac{0.3333333333}{2} \right)} \right] = 0.3552631579$$

$$K_3 = h \left[f \left(t_0 + \frac{h}{2}, y_0 + \frac{K_2}{2} \right) \right] = h \left[\frac{1 + \left(t_0 + \frac{h}{2} \right)}{1 + \left(y_0 + \frac{K_2}{2} \right)} \right] = 0.5 \left[\frac{1 + \left(1 + \frac{0.5}{2} \right)}{1 + \left(2 + \frac{0.3552631579}{2} \right)} \right] = 0.3540372671$$

$$K_4 = hf(t_0 + h, y_0 + K_3) = h \left[\frac{1 + (t_0 + h)}{1 + (y_0 + K_3)} \right] = 0.5 \left[\frac{1 + (1 + 0.5)}{1 + (2 + 0.3540372671)} \right] = 0.3726851852$$

$$\begin{aligned} \text{Thus, } y_1 &= y_0 + \frac{1}{6} [K_1 + 2(K_2 + K_3) + K_4] \\ &= 2 + \frac{1}{6} [0.3333333333 + 2(0.3552631579 + 0.3540372671) + 0.3726851852] \end{aligned}$$

$$y_1 = y(t_0 + h) = y(1.5) = 2.354103228$$

Comparison with the actual values:

Comparison		
Runge-Kutta Third Order Method	Runge-Kutta Fourth Order Method	Actual values
$y(1.5) = 2.354126055$	$y(1.5) = 2.354103228$	$y(1.5) = 2.354101966$

Exercise 1: Numerical Solution of Ordinary Differential Equations

Type-I: Choose the correct answer: (Questions 1-5)

1. For solving the **Initial Value Problem** $\frac{dy}{dt} = f(t, y)$, $t_0 \leq t \leq t_n$, $y(t_0) = y_0$, The **Euler's Method** gives first approximation y_1 as
 - (a) $y_1 = y_0 + hf(t_0, y_0)$
 - (b) $y_2 = y_1 + hf(t_1, y_1)$
 - (c) $y_3 = y_2 + hf(t_2, y_2)$
 - (d) None of these
2. For solving the **Initial Value Problem** $\frac{dy}{dt} = f(t, y)$, $t_0 \leq t \leq t_n$, $y(t_0) = y_0$, The **Third Order Runge-Kutta Method** gives first approximation y_1 as
 - (a) $y_1 = y_0 + \frac{1}{2}[K_1 + 3K_2 + K_3]$
 - (b) $y_1 = y_0 + \frac{1}{6}[K_1 + 4K_2 + K_3]$
 - (c) $y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + K_3]$
 - (d) None of these
3. For solving the **Initial Value Problem** $\frac{dy}{dt} = f(t, y)$, $t_0 \leq t \leq t_n$, $y(t_0) = y_0$, The **Fourth Order Runge-Kutta Method** gives first approximation y_1 as
 - (a) $y_1 = y_0 + \frac{1}{2}[K_1 + 3K_2 + K_3]$
 - (b) $y_1 = y_0 + \frac{1}{6}[K_1 + 4K_2 + K_3]$
 - (c) $y_1 = y_0 + \frac{1}{6}[K_1 + 2(K_2 + K_3) + K_4]$
 - (d) None of these
4. In which of the following Numerical Methods, **derivatives** are required to be calculated for solving **Initial value problem**:
 - (a) Third order Runge-Kutta Method
 - (b) Euler's Method
 - (c) Taylor Series Method
 - (d) None of these
5. **Heun's Method** is a particular case of-
 - (a) Runge-Kutta 4th order method
 - (b) Runge-Kutta 3rd order method
 - (c) Runge-Kutta 2nd order method
 - (d) None of these

Type-I: Solve the following: (Questions 6-16)

6. Use Euler's Method to approximate the solution to the given initial value problem at $t = 1$ and compare the result to the actual values.

$$y' = te^{3t} - 2t, \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad \text{with } h = 0.5$$

Note: Actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

7. Use Euler's Method to approximate the solution to the given initial value problem $t = 2$ and compare the result to the actual values.

$$y' = \frac{1+t}{1+y}, \quad 1 \leq t \leq 2, \quad y(1) = 2, \quad \text{with } h = 0.5$$

Note: Actual solution $y(t) = \sqrt{(t^2 + 2t + 6)} - 1$

8. Use Modified Euler's Method to approximate the solution to the given initial value problem at $t = 2$ and compare the result to the actual values.

$$y' = 1 + \frac{y}{t}, \quad 1 \leq t \leq 2, \quad y(1) = 2, \quad \text{with } h = 0.25$$

Note: Actual solution $y(t) = t \ln t + 2t$

9. Use Taylor's Series Method to approximate the solution to the given initial value problem at $t = 1$ and compare the result to the actual values.

$$y' = te^{3t} - 2t, \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad \text{with } h = 0.5$$

Note: Actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

10. Use Taylor's Series Method to approximate the solution to the given initial value problem at $t = 2$ and compare the result to the actual values.

$$y' = 1 + \frac{y}{t}, \quad 1 \leq t \leq 2, \quad y(1) = 2, \quad \text{with } h = 0.25$$

Note: Actual solution $y(t) = t \ln t + 2t$

11. Use Huen's Method to approximate the solution to the given initial value problem at $t = 1$ and compare the result to the actual values.

$$y' = te^{3t} - 2t, \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad \text{with } h = 0.5$$

Note: Actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

12. Use Huen's Method to approximate the solution to the given initial value problem at $t = 2$ and compare the result to the actual values.

$$y' = 1 + \frac{y}{t}, \quad 1 \leq t \leq 2, \quad y(1) = 2, \quad \text{with } h = 0.25$$

Note: Actual solution $y(t) = t \ln t + 2t$

- 13.** Use Runge-Kutta Third Order Method to approximate the solution to the given initial value problem at $t = 1$ and compare the result to the actual values.

$$y' = te^{3t} - 2t, \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad \text{with } h = 0.5$$

Note: Actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

- 14.** Use Runge-Kutta Third Order Method to approximate the solution to the given initial value problem at $t = 1.5$ and compare the result to the actual values.

$$y' = 1 + \frac{y}{t}, \quad 1 \leq t \leq 2, \quad y(1) = 2, \quad \text{with } h = 0.25$$

Note: Actual solution $y(t) = t \ln t + 2t$

- 15.** Use Runge-Kutta Fourth Order Method to approximate the solution to the given initial value problem at $t = 1$ and compare the result to the actual values.

$$y' = te^{3t} - 2t, \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad \text{with } h = 0.5$$

Note: Actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

- 16.** Use Runge-Kutta Fourth Order Method to approximate the solution to the given initial value problem at $t = 1.5$ and compare the result to the actual values.

$$y' = 1 + \frac{y}{t}, \quad 1 \leq t \leq 2, \quad y(1) = 2, \quad \text{with } h = 0.25$$

Note: Actual solution $y(t) = t \ln t + 2t$

Numerical Analysis-1

Homework-1

Last date: Apr 08, 2023

MATH-316

[For All Campuses]

Max Marks: 05

1. Choose the correct answer (Questions 1.1-1.9)

1.1 Which of the following is a **system of linear equations**:

- (a) $x^3 - 2x^2 - 5 = 0$
- (b) $\begin{cases} \sin x - y = -0.9793 \\ \cos y - x = -0.6703 \end{cases}$
- (c) $\begin{cases} 3x - 10y = 7 \\ 7x + 5y = 4 \end{cases}$
- (d) None of these

1.2 When **Gauss-Jordan method** is used to solve the matrix equation $AX = B$, then the coefficient matrix **A** is transformed into the:

- (a) Upper triangular matrix
- (b) Diagonal matrix
- (c) Zero matrix
- (d) None of these

1.3 Which of the following formula is regarded as the **Percentage Approximate Error (E_P)**:

- (a) $|P_{New} - P_{Old}|$
- (b) $\frac{|P_{New} - P_{Old}|}{|P_{New}|}$
- (c) $\frac{|P_{New} - P_{Old}|}{|P_{New}|} \cdot 100$
- (d) None of these

1.4 A real root of the equation: $2x - \log_{10}(x) = 7$ accurate to within 10^{-4} is

- (a) 1.78927
- (b) 2.78927
- (c) 3.78927
- (d) None of these

1.5 The interval for equation $x^{22} = 69$, in which at least one real root exists, given by:

- (a) $[6, 7]$
- (b) $[0, 1]$
- (c) $[-7, -6]$
- (d) None of these

1.6 The solution of a linear system of equations: $4x + y + 3z = 17$, $x + 5y + z = 14$, $2x - y + 8z = 12$ by **Gauss Jacobi Method** for first iteration (Starting with $x^{(0)} = 1$, $y^{(0)} = 1$ & $z^{(0)} = 1$) is

- (a) $x=4.25, y=2.8, z=1.5$
- (b) $x=4, y=2.0, z=1.0$
- (c) $x=4.25, y=1.66, z=1.1$
- (d) $x=3.25, y=2.40, z=1.375$

1.7 Which of the following **numerical methods** provide with the **Exact Solutions** for a given **linear system of algebraic equations**:

- (a) Gauss-Elimination method
- (b) Gauss-Jordan method
- (c) LU-Factorization method
- (d) All of these

1.8 The **Back Substitution Method** provides solution for the given **Matrix Equation**-

$$\begin{bmatrix} 1 & -4 & -2 \\ 0 & 5 & 14 \\ 0 & 0 & -253/5 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -20 \\ 43 \\ -506/5 \end{bmatrix} \text{ as follows:}$$

- (a) $u = 4, v = -3, w = 2$
- (b) $u = -4, v = 3, w = 2$
- (c) $u = 4, v = 3, w = -2$
- (d) None of these

1.9 A **real root** of the equation: $x^2 = e^{(\frac{x}{2})}$, accurate to within 10^{-5} is

- (a) 1.99457
- (b) 4.56702
- (c) 5.42961
- (d) None of these

2. Find the interval $[a, b]$ in which, the given equation $f(x) = 0$ has at least one real root.

- 2.1 $x^3 + 4x^2 - 10 = 0$ Interval: [,]
- 2.2 $\sin x - e^{-x} = 0$ Interval: [,]
- 2.3 $x^3 - 2x^2 - 5 = 0$ Interval: [,]
- 2.4 $2x - \log_{10}(x) = 12$ Interval: [,]

3. Solve the following **linear system of equations**:

$$3.1 \quad \left. \begin{array}{l} 2x + 4y + z = 3 \\ 3x + 2y - 2z = -2 \\ x - y + z = 6 \end{array} \right\} \text{ by using Gauss-Jordan Method}$$

$$3.2 \quad \left. \begin{array}{l} 2u - 3v + 4w = 13 \\ 3u + 4v + 5w = 40 \\ u + v + w = 9 \end{array} \right\} \text{ by using Gauss-Elimination Method}$$

$$3.3 \quad \left. \begin{array}{l} 5x - 2y + z = 4 \\ 7x + y - 5z = 8 \\ 3x + 7y + 4z = 10 \end{array} \right\} \text{ by using LU-Factorization Method.}$$

$$3.4 \quad \left. \begin{array}{l} x + 5y + z = 14 \\ 2x - y + 8z = 12 \\ 4x + y + 3z = 17 \end{array} \right\} \text{ by using Gauss-Seidel Method up to three iterations}$$

(Starting with $x^{(0)} = 1, y^{(0)} = 1, z^{(0)} = 1$)

$$3.5 \quad \left. \begin{array}{l} 4\alpha - \beta + \gamma = 8 \\ 2\alpha + 5\beta + 2\gamma = 3 \\ \alpha + 2\beta + 4\gamma = 11 \end{array} \right\} \text{ by Gauss- Jacobi iterative technique up to three iterations}$$

(Starting with $x^{(0)} = 1, y^{(0)} = 1, z^{(0)} = 1$)

4. Answer the following Questions and Calculate Relative Error and Percentage Error in each case:

4.1 Use Bisection Method to find a solution of the equation $3x - e^x = 0$ on the interval $[1, 2]$ with an accuracy of 10^{-3} .

4.2 Use Newton-Raphson Method to find a solution accurate to within 10^{-6} for the following equation: $\sin x = e^{-x}$, for $0 \leq x \leq 1$

4.3 Use Secant Method to find an approximation to $(29)^{1/4}$, accurate to within 10^{-4}

5. Use Newton-Iterative Techniques to find approximate solutions up to 2nd iterations to the given nonlinear system of equations. [initial approximation $(x_0, y_0) = (0.75, 0.75)$]

$$\left. \begin{array}{l} x^2 + y = 11 \\ x + y^2 = 7 \end{array} \right\}$$

Numerical Analysis-1

Homework-2

Last date: May 18, 2023

MATH-316

[For All Campuses]

Max Marks: 05

1. Choose the correct answer (Questions 1.1 -1.8)

1.1 Identify the given **Rule**: $\int_{x_0}^{x_n} f(x) dx = \frac{3h}{8} \{f(x_0) + 2[f(x_3) + f(x_6) + f(x_9) + \dots + f(x_{n-3})] + 3[f(x_1) + f(x_2) + f(x_4) + f(x_5) + \dots + f(x_{n-1})] + f(x_n)\}$

- (a) Trapezoidal Rule
- (b) Simpson's $\frac{3}{8}$ Rule
- (c) Newton-Raphson Rule
- (d) None of these

1.2 If $x_0 = 2$, $x_1 = 3$ **and** $x_2 = 5$, **then Find** the **Lagrange's interpolating polynomial** of degree **two** for $f(x) = \frac{1}{x}$:

- (a) $\frac{1}{30}x^2 + \frac{1}{3}x - \frac{31}{30}$
- (b) $\frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$
- (c) $\frac{1}{30}x^2 - \frac{1}{3}x + \frac{31}{30}$
- (d) $\frac{1}{22}x^2 + \frac{35}{88}x - \frac{49}{44}$

1.3 The relation between **operators**: $\mu^2 - 1 \equiv$

- (a) $\frac{1}{4}(E + E^{-1} - 2)$
- (b) $\frac{1}{4}\delta^2$
- (c) $\frac{1}{4}\Delta\nabla$
- (d) All of these

1.4 Simpson's $\frac{3}{8}$ rule approximates $\int_0^1 \frac{dx}{1+x^2} = 0.785395862$, (with $n = 6$), Hence **find the value** of π :

- (a) $\pi=3.141583448$
- (b) $\pi=3.141592654$
- (c) $\pi=3.141568600$
- (d) None of these

1.5 Given: $\Delta f(1.0) = 4.6$, $\Delta^2 f(1.0) = -0.2$, $\Delta^3 f(1.0) = 0.1$, $\Delta^4 f(1.0) = 0.1$ **and** step-size $h = 0.1$, **then the value of** $f''(1.0)$ **is equal to:**

- (a) 47.08333333
- (b) -20.8333333
- (c) 2.08333333
- (d) None of these

1.6 If $f(1.5) = -0.25$, $f(3) = 2$ and $f(6) = 20$ then **Newton's Divided Difference Interpolation Formula** will approximate $f(4)$ by

- (a) -6
- (b) 6
- (c) 12
- (d) -12

1.7 The **relation** between **operators**: $\delta - 2\mu \equiv$

- (a) $2E^{\frac{1}{2}}$
- (b) $-2E^{\frac{1}{2}}$
- (c) $2E^{-\frac{1}{2}}$
- (d) $-2E^{-\frac{1}{2}}$

1.8 If $f(0) = 1$, $f(1) = 2$, $f(2) = 1$ and $f(3) = 10$ then **Lagrange's Interpolation Polynomial** will approximate $f(2.5)$ by:

- (a) 2.5
- (b) 3.5
- (c) -2
- (d) None of these

2. Prove with the **usual notations** that:

2.1 $\Delta \equiv \delta E^{\frac{1}{2}}$

2.2 $\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)(1 + \Delta)^{\frac{1}{2}} \equiv 2 + \Delta$

2.3 $\Delta + \nabla \equiv \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

3. Solve the following (**Questions 3.1 -3.3**)

3.1 Use **Lagrange Interpolating Polynomial Formula** to approximate $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$

3.2 Use **Newton Divided Difference Interpolating Polynomial Formula** to find a polynomial function $f(x)$ from the given data-

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

Hence find: (i) $f(1)$ (ii) $f'(1)$ (iii) $f'(2)$ (iv) $f''(3)$

3.3 Use **Lagrange Inverse Interpolating Polynomial Formula** to find the value of x when $f(x) = 100$ from the following data.

x	0	1	2	3
$f(x)$	0	20	60	120

4. Answer the following (Questions 4.1 -4.5)

4.1 Approximate the integral $\int_0^{\pi} x \cos(x^2) dx$ using **Simpson's $\frac{3}{8}$** rule with **$n = 6$** .

4.2 Approximate the integral $\int_0^{\frac{\pi}{2}} \sqrt{\cos x} dx$ using **Simpson's $\frac{1}{3}$** rule with **$h = \frac{\pi}{12}$**

4.3 Evaluate the integral $\int_3^5 \frac{1}{\sqrt{x^2-4}} dx$ by using **Trapezoidal rule** with **$n = 5$**

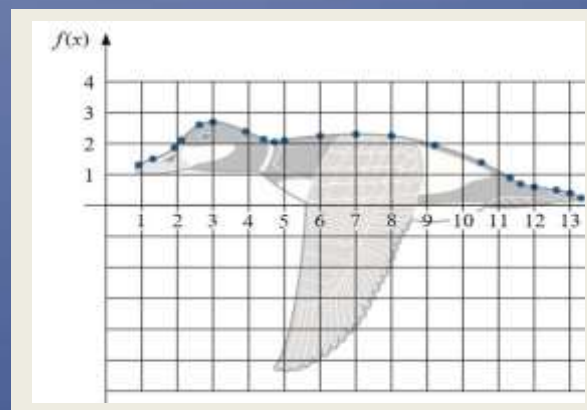
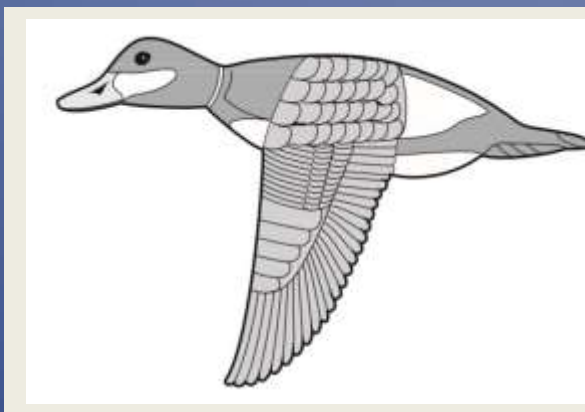
4.4 Use **Direct Formulae for Derivatives** to approximate **$f'(3)$** and **$f''(1)$** from the following table.

x	1	3	5	7	9
$f(x)$	85.3	74.5	67.0	60.5	54.3

4.5 Use the **Direct Formulae for Derivatives** to approximate **$f'(2.0)$** , **$f''(2.0)$** and **$f'(1.8)$** from the given data:

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891

Numerical Analysis



Lecture Notes: Math-316 (Science College)

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