



جامعة الملك عبدالعزيز
KING ABDULAZIZ UNIVERSITY

Discrete Structure– COCS 222

<Logic>



Faculty of Computing & Information
Technology- Rabigh
King Abdulaziz University

Introduction

What's wrong with this proof ?

If you figure it out, do NOT say the answer –
let others think about it as well.

- | | | |
|----|---|-----------------------------|
| 1. | Let a and b be non-zero such that | $a = b$ |
| 2. | Multiply both sides by a | $a^2 = ab$ |
| 3. | Subtract b^2 | $a^2 - b^2 = ab - b^2$ |
| 4. | Factor both sides | $(a - b)(a + b) = b(a - b)$ |
| 5. | Divide by $(a - b)$ | $a + b = b$ |
| 6. | Since $a = b$, we replace b with a | $b + b = b$ |
| 7. | Combine terms | $2b = b$ |
| 8. | As b is non-zero, divide it out | $2 = 1$ |

Introduction

Why study Discrete Mathematics?

- Develop Mathematical maturity.
- Discrete Mathematics provides the mathematical foundations for many computer science courses including data structures, algorithms, database theory, computer security, operating systems, and more...
- Discrete Mathematics contains the necessary mathematical background for solving problems in many other domains:
operations research, chemistry, biology, etc

Introduction

•Is this a hard class? Many students find it challenging significantly more challenging than previous math courses

Why is it more challenging?

Main reason : Primary goal of the course is to teach mathematical reasoning and problem solving

We want to teach you how to solve problems and how to logically think about and find solutions

Introduction

■ Proofs:

- How do you prove something is correct?
- Maybe you are arguing with someone...
 - How can you formulate (come up with) a logically sound (logically correct) proof, showing that you are correct?
- How do you know when something is not correct?
 - Such as showing that $2=1$?
- How do you think logically?
- How do you think to solve problems?

Introduction

If you figure it out, do NOT say the answer –
let others think about it as well.

- | | | |
|----|---|------------------------|
| 1. | Let a and b be non-zero such that | $a = b$ |
| 2. | Multiply both sides by a | $a^2 = ab$ |
| 3. | Subtract b^2 | $a^2 - b^2 = ab - b^2$ |
| 4. | Factor both sides | $(a-b)(a+b) = b(a-b)$ |
| 5. | Divide by $(a-b)$ | $a+b = b$ |
| 6. | Since $a = b$, we replace b with a | $b+b = b$ |
| 7. | Combine terms | $2b = b$ |
| 8. | As b is non-zero, divide it out | $2 = 1$ |

Q.E.D. (Latin for “which was to be proven”)

Introduction

- Logical Thinking & Reasoning
 - This is a MAJOR part of this course
 - Training your brain to use reason and logic to formulate/create sound (good) arguments
- The following slide shows an example of a problem.
- There is a mistake in the logic
 - But it is hard to find the mistake
- See if you can identify the mistake in the logic.

Introduction

Three people check into a hotel. They pay 300sr (total) to the manager, and then they go to their room. The manager suddenly remembers that the price for the room is only 250sr (meaning, the guests paid too much money). So he gives 50sr to a worker, and he tells the worker to return the 50sr to the people.

On the way to the room, the worker realizes that 50sr would be difficult to share among three people ($50/3$ does not divide evenly). So he decides to keep 20sr (puts it in his pocket!), and then he gives 10sr to each person.

Now, each person paid 100sr and got back 10sr. So they paid 90sr each, totaling 270sr. The worker has 20sr in his pocket, totaling 290sr.
Where is the missing 10sr?

Let's get started with...

Logic!

Logic

- Crucial for mathematical reasoning
- Important for program design
- Used for designing electronic circuitry
- (Propositional)Logic is a system based on **propositions**.
- A proposition is a (declarative) statement that is either **true** or **false** (not both).
- We say that the **truth value** of a proposition is either true (**T**) or false (**F**).
- Corresponds to **1** and **0** in digital circuits

The Statement/Proposition Game

“Elephants are bigger than mice.”

Is this a statement? *yes*

Is this a proposition? *yes*

What is the truth value
of the proposition? *true*

The Statement/Proposition Game

"520 < 111"

Is this a statement? *yes*

Is this a proposition? *yes*

What is the truth value
of the proposition? *false*

The Statement/Proposition Game

“ $y > 5$ ”

Is this a statement? *yes*

Is this a proposition? *no*

Its truth value depends on the value of y , but this value is not specified.

We call this type of statement a propositional function or open sentence.

The Statement/Proposition Game

“Today is January 27 and $99 < 5$.”

Is this a statement? *yes*

Is this a proposition? *yes*

What is the truth value
of the proposition? *false*

The Statement/Proposition Game

“Please do not fall asleep.”

Is this a statement? **no**

It's a request.

Is this a proposition? **no**

Only statements can be propositions.

The Statement/Proposition Game

“If the moon is made of cheese, then I will be rich.”

Is this a statement?

yes

Is this a proposition?

yes

What is the truth value
of the proposition?

probably true

The Statement/Proposition Game

“ $x < y$ if and only if $y > x$.”

Is this a statement? *yes*

Is this a proposition? *yes*

... because its truth value does not depend on specific values of x and y .

What is the truth value of the proposition? *true*

Combining Propositions

As we have seen in the previous examples, one or more propositions can be combined to form a single **compound proposition**.

We formalize this by denoting propositions with letters such as **p, q, r, s**, and introducing several **logical operators or logical connectives**.

P = Bander attended the last class

$\neg p$ =

Logical Operators (Connectives)

We will examine the following logical operators:

- Negation (NOT, $\neg$)
- Conjunction (AND, $\wedge$)
- Disjunction (OR, $\vee$)
- Exclusive-or (XOR, $\oplus$)
- Implication (if – then, $\rightarrow$)
- Biconditional (if and only if, $\leftrightarrow$)

Truth tables can be used to show how these operators can combine propositions to compound propositions.

Negation (NOT)

Unary Operator, Symbol: $\neg$

P	$\neg P$
true (T)	false (F)
false (F)	true (T)

Conjunction (AND)

Binary Operator, Symbol: $\wedge$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (OR)

Binary Operator, Symbol: $\vee$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or (XOR)

Binary Operator, Symbol: $\oplus$

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication (if - then)

Binary Operator, Symbol: $\rightarrow$

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$\neg Q \rightarrow \neg P$	$\neg Q$	$\neg P$
T	T	T	T	T	F	F
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Converse, Inverse, and Contrapositive of a Conditional Statement

We can form some new conditional statements starting with a conditional statement $p \rightarrow q$.

In particular, there are three related conditional statements that occur so often that they have special names.

The **converse** of $p \rightarrow q$ is the proposition $q \rightarrow p$

The **inverse** of $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$

The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$

We will see that of these three conditional statements formed from $p \rightarrow q$, only the contrapositive always has the same truth value as $p \rightarrow q$.

Biconditional (if and only if)

Binary Operator, Symbol: $\leftrightarrow$

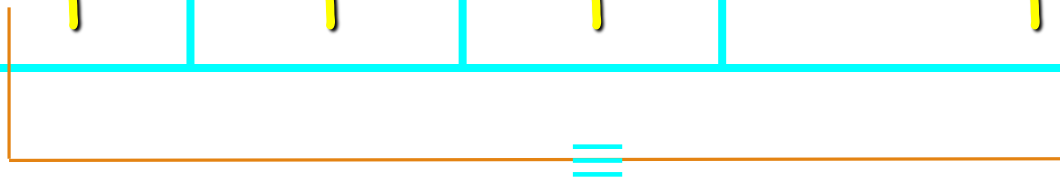
P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Using the truth table prove that $P \leftrightarrow Q$ is equivalent to $(p \rightarrow Q)$ and $(Q \rightarrow p)$?

Biconditional (if and only if)

Binary Operator, Symbol: $\leftrightarrow$

P	Q	$P \leftrightarrow Q$	$p \Rightarrow Q$	$Q \Rightarrow p$	$(p \Rightarrow Q) \wedge (Q \Rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T



Statements and Operators

Statements and operators can be combined in any way to form new statements.

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Statements and Operations

Statements and operators can be combined in any way to form new statements.

P	Q	$P \wedge Q$	$\neg (P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Statements and Operations

Statements and operators can be combined in any way to form new statements.

P	Q	$P \wedge Q$	$\neg (P \vee Q)$	$(\neg P) \wedge (\neg Q)$
T	T			
T	F			
F	T			
F	F			

Precedence of Logical Operators

TABLE 8

**Precedence of
Logical Operators.**

<i>Operator</i>	<i>Precedence</i>
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Exercises

- To take discrete mathematics, you must have taken calculus or a course in computer science.

- P: take discrete mathematics
- Q: take calculus
- R: take a course in computer science

$$P \rightarrow Q \vee R$$

Exercises

- When you buy a new car from Toyota Motor Company, you get SAR2000 back in cash or a 2% car loan.
 - P: buy a car from Toyota Motor Company
 - Q: get SAR2000 cash back
 - R: get a 2% car loan

$$P \rightarrow Q \oplus R$$

Why use XOR here? – example of *ambiguity of natural languages*

Exercises

- School is closed if more than 2 meters of snow falls or if the wind chill is below -100.

- P: School is closed
- Q: 2 meters of snow falls
- R: wind chill is below -100

$$(Q \vee R) \rightarrow P$$

- Precedence among operators: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

Equivalent Statements

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

The statements $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are **logically equivalent**, since they have the same truth table, or put it in another way, $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$ is always true.

Tautologies and Contradictions

A tautology is a statement that is always true.

Examples:

- $R \vee (\neg R)$
- $\neg (P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

A contradiction is a statement that is always false.

Examples:

- $R \wedge (\neg R)$
- $\neg (\neg (P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q))$

The negation of any tautology is a contradiction, and the negation of any contradiction is a tautology.

Tautology and Contradiction

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.

A compound proposition that is always false is called a contradiction.

Example

Can we construct examples of tautologies and contradictions using just one propositional variable?

Solution:

We can See table 1

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Because $p \vee \neg p$ is always true, it is a tautology.

Because $p \wedge \neg p$ is always false, it is a contradiction

Contingency

Definition: A statement that can be either true or false, depending on the truth values of its propositional variables, is called a **contingency**.

Example: The statement $(p \Rightarrow q) \wedge (p \vee q)$ is a Contingency

p	q	$p \Rightarrow q$	$p \vee q$	$(p \Rightarrow q) \wedge (p \vee q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

Equivalence

Definition: two propositional statements $S1$ and $S2$ are said to be (logically) equivalent, denoted $S1 \equiv S2$ if

- They have the same truth table, or
- $S1 \Leftrightarrow S2$ is a tautology

Equivalence can be established by

- Constructing truth tables
- Using equivalence laws (Table 6 in next slid)

Equivalence

Logical Equivalence

$$P \rightarrow q$$

is it equivalent to

$$q \rightarrow p ???$$

TABLE 6 Logical Equivalences.

<i>Equivalence</i>	<i>Name</i>
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Equivalence

Equivalence laws

- Identity laws, $P \wedge T \equiv P,$
- Domination laws, $P \wedge F \equiv F,$
- Idempotent laws, $P \wedge P \equiv P,$
- Double negation law, $\neg(\neg P) \equiv P$
- Commutative laws, $P \wedge Q \equiv Q \wedge P,$
- Associative laws, $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R,$
- Distributive laws, $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R),$
- De Morgan's laws, $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$
- Law with implication $P \rightarrow Q \equiv \neg P \vee Q$

Exercises

Show that $P \rightarrow Q \equiv \neg P \vee Q$: by truth table

Solution:-

P	Q	$P \rightarrow Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Exercises

Show that $(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)$: by truth table

Solution:-

P	Q	R	$P \rightarrow Q$	$P \rightarrow R$	$(P \rightarrow Q) \wedge (P \rightarrow R)$	$(Q \wedge R)$	$P \rightarrow (Q \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Exercises

Show that $(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)$:
by equivalence laws.

Solution:-

$$\begin{aligned}\text{LHS} &\equiv (P \rightarrow Q) \wedge (P \rightarrow R) \\ &\equiv (\neg P \vee Q) \wedge (\neg P \vee R) \\ &\equiv \neg P \vee (Q \wedge R) \\ &\equiv P \rightarrow (Q \wedge R), \text{ Since if } \neg P \vee X \text{ then } P \rightarrow X \\ &\equiv \text{RHS}\end{aligned}$$

Exercises

Show that $(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)$:

by equivalence laws

Solution:-

$$\begin{aligned} \text{RHS} &\equiv P \rightarrow (Q \wedge R) \\ &\equiv \neg P \vee (Q \wedge R) \\ &\equiv (\neg P \vee Q) \wedge (\neg P \vee R) \\ &\equiv (P \rightarrow Q) \wedge (P \rightarrow R) \\ &\equiv \text{LHS} \end{aligned}$$

Exercises

Show that: $P \rightarrow (Q \wedge R) \equiv (P \rightarrow Q) \wedge (P \rightarrow R)$, by equivalence laws

Solution:-

$$\text{LHS} \equiv P \rightarrow (Q \wedge R)$$

$$\equiv \neg P \vee (Q \wedge R), \text{ (Law with implication)}$$

$$\equiv (\neg P \vee Q) \wedge (\neg P \vee R), \text{ (Distribution law)}$$

$$\equiv (P \rightarrow Q) \wedge (P \rightarrow R), \text{ (Law with implication)}$$

$$\equiv \text{RHS}$$

Exercises

Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent?

We could use a truth table to show that these compound propositions are equivalent.

Or

Solution: -

$$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q), \quad (\text{by implication law})$$

$$\equiv \neg(\neg p) \wedge \neg q, \quad (\text{by the second De Morgan law})$$

$$\equiv p \wedge \neg q, \quad (\text{by the double negation law})$$

Exercises

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology?

Solution:-

$$(p \wedge q) \rightarrow (p \vee q) \equiv \neg (p \wedge q) \vee (p \vee q), \text{ (Law with implication)}$$

$$\equiv (\neg p \vee \neg q) \vee (p \vee q), \text{ (by the first De Morgan law)}$$

$$\equiv (\neg p \vee p) \vee (\neg q \vee q)$$

(by the associative and commutative laws for disjunction)

$$\equiv (T \vee T), \text{ (by the negation law)}$$

$$\equiv T, \text{ (by the domination law)}$$

(Note: This could also be done using a truth table.)

Propositional Functions & Predicates

Propositional function (open sentence):

statement involving one or more variables, **e.g. : $x-3 > 5$.**

Let us call this propositional function **$P(x)$** , where **P** is the **predicate** and **x** is the **variable**.

What is the truth value of $P(2)$? **false**

What is the truth value of $P(8)$? **false**

What is the truth value of $P(9)$? **true**

When a variable is given a value, it is said to be instantiated

Truth value depends on value of variable

Propositional Functions

Let us consider the propositional function $Q(x, y, z)$ defined as:

$$x + y = z.$$

Here, Q is the *predicate* and x , y , and z are the *variables*.

What is the truth value of $Q(2, 3, 5)$? *true*

What is the truth value of $Q(0, 1, 2)$? *false*

What is the truth value of $Q(9, -9, 0)$? *true*

A propositional function (predicate) becomes a proposition when *all* its variables are *instantiated*.

Propositional Functions

Other examples of propositional functions

Person(x), which is true if **x** is a **person**

Person(Socrates) = T, **Person(dolly-the-sheep) = F**

CSCourse(x), which is true if **x** is a computer science course

CSCourse(CMSC201) = T, **CSCourse(MATH155) = F**

How do we say:

- All humans are mortal
- One CS course

Universal Quantification

Let $P(x)$ be a predicate (propositional function).

Universally quantified sentence:

For all x in the **universe of discourse** $P(x)$ is true.

Using the universal quantifier $\forall$:

$\forall x P(x)$ ***“for all x $P(x)$ ” or “for every x $P(x)$ ”***

(Note: $\forall x P(x)$ is either true or false, so it is a proposition, not a propositional function.)

Universal Quantification

Example: Let the universe of discourse be all people

$S(x)$: x is a KAU student.

$G(x)$: x is a genius.

What does **$\forall x (S(x) \rightarrow G(x))$** mean ?

“If x is a KAU student, then x is a genius.” or

“All KAU students are geniuses.”

If the universe of discourse is all KAU students, then the same statement can be written as **$\forall x G(x)$**

Existential Quantification

Existentially quantified sentence:

There exists an x in the universe of discourse for which $P(x)$ is true.

Using the existential quantifier $\exists$:

$\exists x P(x)$ “There is an x such that $P(x)$.”

“There is at least one x such that $P(x)$.”

(Note: $\exists x P(x)$ is either true or false, so it is a proposition, but no propositional function.)

Existential Quantification

Example:

$P(x)$: x is a KAU professor.

$G(x)$: x is a genius.

What does $\exists x (P(x) \wedge G(x))$ mean ?

“There is an x such that x is a KAU professor and x is a genius.”

or

“At least one KAU professor is a genius.”

Quantification

Another example:

Let the universe of discourse be the real numbers.

What does $\forall x \exists y (x + y = 320)$ mean ?

“For every x there exists a y so that $x + y = 320$.”

Is it true?

yes

Is it true for the natural numbers?

no

Disproof by Counterexample

A counterexample to $\forall x P(x)$ is an object c so that $P(c)$ is false.

Statements such as $\forall x (P(x) \rightarrow Q(x))$ can be disproved by simply providing a counterexample.

Statement: "All birds can fly."

Disproved by counterexample: Penguin.

Negation

$\neg(\forall x P(x))$ is logically equivalent to $\exists x (\neg P(x))$.

$\neg(\exists x P(x))$ is logically equivalent to $\forall x (\neg P(x))$.

This is de Morgan's law for quantifiers

Negation

Examples

Not all roses are red

$$\begin{aligned}\neg \forall x (\text{Rose}(x) \rightarrow \text{Red}(x)) &\rightarrow \neg(\neg \text{Rose}(x) \vee \text{Red}(x)) \\ &\rightarrow (\text{Rose}(x) \vee \neg \text{Red}(x))\end{aligned}$$

Hence, $\exists x (\text{Rose}(x) \wedge \neg \text{Red}(x))$

Nobody is perfect

$$\begin{aligned}\neg \exists x (\text{Person}(x) \wedge \text{Perfect}(x)) \\ \forall x (\text{Person}(x) \rightarrow \neg \text{Perfect}(x))\end{aligned}$$

Nested Quantifier

A predicate can have more than one variables.

- $S(x, y, z)$: z is the sum of x and y
- $F(x, y)$: x and y are friends

We can quantify individual variables in different ways

- $\forall x, y, z (S(x, y, z) \rightarrow (x \leq z \wedge y \leq z))$
- $\exists x \forall y \forall z (F(x, y) \wedge F(x, z) \wedge (y \neq z) \rightarrow \neg F(y, z))$

Nested Quantifier

Exercise: translate the following English sentence into logical expression

“There is a rational number in between every pair of distinct rational numbers”

Use predicate $Q(x)$, which is true when x is a rational number

$$\forall x, y (Q(x) \wedge Q(y) \wedge (x < y) \rightarrow \exists u (Q(u) \wedge (x < u) \wedge (u < y)))$$