



353PHYS - 20222

STATISTICAL PHYSICS



DR. LAURA MUQRI
ASSISTANT PROFESSOR
PHYSICS DEPARTMENT
SCIENCE COLLEGE-JU

GRADE DISTRIBUTION

Total	Final Exam	Theoretical			Grade Distribution
	Theoretical	Quizzes	HWs	Midterms	
100	50	5	15	30	Marks

*** Two Quizzes and Five Homework Sets**

COURSE LEARNING OUTCOMES

Code	Course Learning Outcomes	Program Learning Outcomes
Knowledge		
CLO1.1	Describe the statistical nature of concepts and laws in thermodynamics,	PLO1.1
CLO1.2	Define statistical function, such as Boltzmann distribution, Gibbs distribution, Fermi-Dirac and Bose-Einstein distributions to solve problems in some physical systems	PLO1.1
CLO1.3	Explain the fundamental postulates of statistical mechanics	PLO1.2
Skills		
CLO2.1	Calculate statistical properties for systems such as gases, solids, photons or vibrations.	PLO2.1
CLO2.2	Compare between statistical laws of classical particles, bosons and fermions	PLO2.2
CLO2.3	Develop communication and critical thinking competencies during interactive discussion, group assignments, essays or web-based activities	PLO2.4
Values		
CLO3.1	Develop skills of working in groups in group assignments and discussion and bear individual responsibility in the assigned tasks	PLO3.1

COURSE CONTENT

No.	List of Topics	Contact Hours
1	Introduction to Statistical Physics	6
2	Statistical description of systems of particles (Methodology of Statistical Mechanics)	8
3	Classical Statistical Mechanics	8
4	Quantum Statistical Mechanics	6
5	Review	2
Total		30

LEARNING RESOURCES

Required Textbook	Fundamentals of Statistical and Thermal Physics ; F. Reif, McGraw–Hill, 2002 (1965)
Essential References	Thermodynamics, Kinetic Theory and Statistical Thermodynamics; F.W.Sears and G. L Salinger, John Wiley& Sons, Inc., 1975. Introduction to Statistical Physics, W. G. Rosswe, Ellis Horwood, Ltd.1982
Electronic Learning Resources	Assigned during lectures
Other Learning Materials	Given in class

IMPORTANT!

- You must keep track of mathematical methods, definitions and statistical methods (devote a section to each in your notebook)
- Submit your homework on or before its due date.
- Before attending class; revise the previous lecture, read the upcoming section from the textbook and PARTICIPATE!

A GENTLE REMINDER!

akmuqri@jazanu.edu.sa

Name + University ID #+ Section #

CHAPTER ONE

INTRODUCTION TO STATISTICAL METHODS

09/05/2021 & 09/07/2021

1. INTRODUCTION TO STATISTICAL METHODS

Random Walk and Binomial Distribution

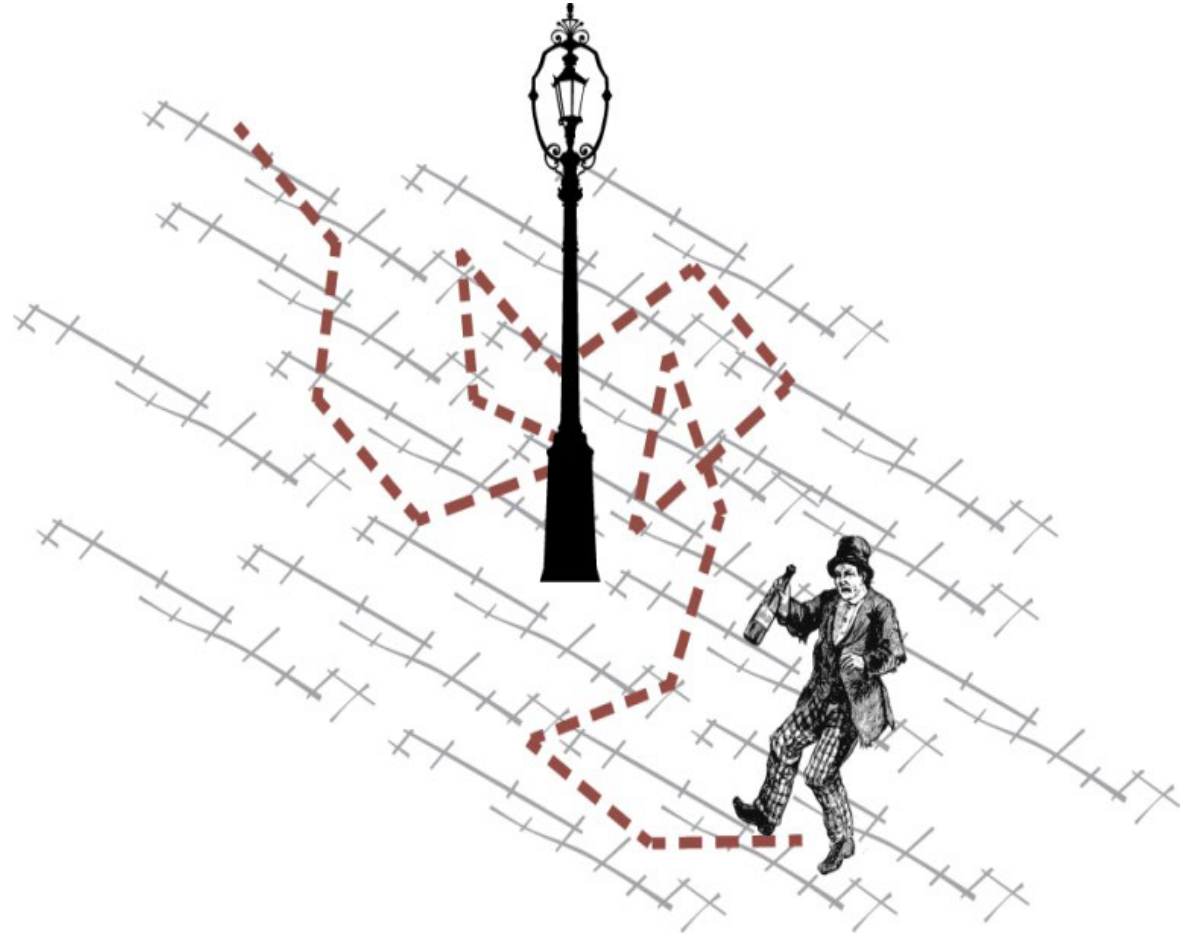
- 1.1 Elementary Statistical Concepts & Examples
- 1.2 The Simple Random Walk Problem In One-dimension
- 1.3 General Discussion of Mean Values
- 1.4 Calculation of Mean Values for the Random Walk Problem
- 1.5 Probability Distribution for Large N
- 1.6 Gaussian Probability Distributions

1.1 ELEMENTARY STATISTICAL CONCEPTS & EXAMPLES

- It is important to keep in mind that when describing a situation from a statistical point of view (i.e., in terms of probabilities) it is important to consider an assembly or (an ensemble).
- The probability of occurrence is then defined with respect to this ensemble and is given by the fraction of systems in the ensemble which are characterized by the occurrence of this specified event.

1.1 ELEMENTARY STATISTICAL CONCEPTS & EXAMPLES

An ideal, illustrative example
of basic probability concepts is
the “Random-walk Problem”
casually called “Drunkard’s
Walk Problem”



1.1 ELEMENTARY STATISTICAL CONCEPTS & EXAMPLES

- Since each step is of length l , the location of the man must be of the form $x = ml$ where m is an integer (positive, negative, zero)
- The question of interest is then:

After the man has taken N steps what is the
PROBABILITY of him being located at $x = ml$?

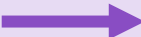
- The main problem is adding N vectors of the same length but of random directions and then asking for the probability that their vector sum has a certain magnitude & direction.

1.2 THE SIMPLE RANDOM WALK PROBLEM IN ONE-DIMENSION

- Vocabulary change : particles..
- After a total of N successive steps, each of length l , the particle is located at $x = ml$
- Where m is an integer lying between $-N \leq m \leq N$
- We want to calculate the probability $P_N(m)$ of finding the particle at position $x = ml$ after N such steps.
- Let n_1 be the number of steps to the right and n_2 be the number of steps to the left then the total number of steps N is given by $N = n_1 + n_2$
- The net displacement m measured to the right in units of step length is $m = n_1 - n_2$

$$m = n_1 - n_2 = n_1 - (N - n_1) = 2n_1 - N$$

1.2 THE SIMPLE RANDOM WALK PROBLEM IN ONE-DIMENSION

Quantity	Definition
n_1	The number of steps to the right  or (forward)
n_2	The number of steps to the left  or (backward)
$N = n_1 + n_2$	The total number of steps taken in both directions
$m = n_1 - n_2$	The net displacement measured to the right in units of step length The range of it is $-N \leq m \leq N$, $m = 2n_1 - N$
l	The length of each step
$x = ml$	The final position of the particle
p	The probability of a step being to the right
q	The probability of a step being to the left with $p + q = 1$

1.2 THE SIMPLE RANDOM WALK PROBLEM IN ONE-DIMENSION

$$m = n_1 - n_2 = n_1 - (N - n_1) = 2n_1 - N$$

- This shows that if N is odd then the possible values of m must also be odd. Conversely, if N is even, m must also be even.
- Each step is characterized by the respective probability

p = probability that the step is to the right

$q = 1 - p$ = probability that the step is to the left

- The probability of any given sequence of n_1 steps to the right and n_2 steps to the left is given simply by multiplying their respective probabilities

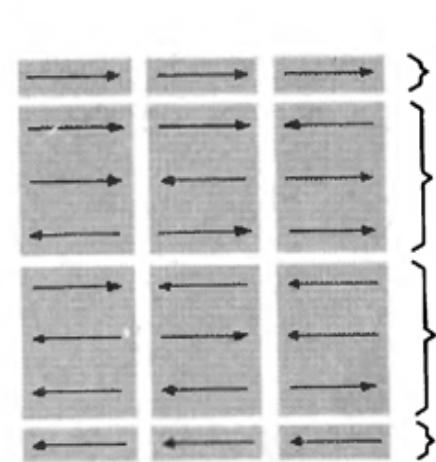
$$\underbrace{pp \dots p}_{n_1 \text{ factors}} \underbrace{qq \dots q}_{n_2 \text{ factors}} = p^{n_1} q^{n_2}$$

1.2 THE SIMPLE RANDOM WALK PROBLEM IN ONE-DIMENSION

Example (1) – page 8

- In the case of a total of **N=3** steps, there is one way such that all successive can be to the right **m=3**
- Probability that all successive steps are to the right

$$W_3(3) = \frac{3!}{3!0!} p^3 q^0 = p \cdot p \cdot p = p^3$$



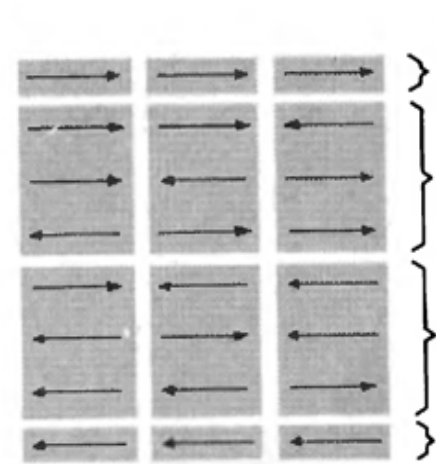
n_1	n_2	m
3	0	3
2	1	1
1	2	-1
0	3	-3

1.2 THE SIMPLE RANDOM WALK PROBLEM IN ONE-DIMENSION

Example (1) – page 8

- In the case of a sequence of two steps to the right and the third step to the left, there are three possible sequences thus the probability for such measurement is

$$W_3(3) = \frac{3!}{2!1!} p^2 q^1 = 3 p \cdot p \cdot q = 3p^2 q$$



n_1	n_2	m
3	0	3
2	1	1
1	2	-1
0	3	-3

1.2 THE SIMPLE RANDOM WALK PROBLEM IN ONE-DIMENSION

The Binomial Distribution Function

$$(p + q)^N = \sum_{n=0}^N \frac{N!}{n! (N-n)!} p^n q^{N-n}$$

Considering the particle's net displacement from the origin m , the probability $P_N(m)$ of finding the particle at a position m after N steps is the same as $W_N(n_1)$

$$P_N(m) = W_N(n_1)$$

One can find that

$$n_1 = \frac{1}{2} (N + m), \quad n_2 = \frac{1}{2} (N - m)$$

$$P_N(m) = W_N(n)$$

1.2 THE SIMPLE RANDOM WALK PROBLEM IN ONE-DIMENSION

Rewriting $P_N(m)$ in terms of n_1 & n_2 we have

$$P_N(m) = \frac{N!}{[(N+m)/2]![(N-m)/2]!} p^{(N+m)/2} q^{(N-m)/2}$$

$$P_N(m) = \frac{N!}{[(N+m)/2]![(N-m)/2]!} p^{(N+m)/2} (1-p)^{(N-m)/2} \quad (\text{knowing that } p + q = 1)$$

In the special case where $p = q = \frac{1}{2}$

$$P_N(m) = \frac{N!}{[(N+m)/2]![(N-m)/2]!} \left(\frac{1}{2}\right)^{(N+m)/2} \left(\frac{1}{2}\right)^{(N-m)/2} = \frac{N!}{[(N+m)/2]![(N-m)/2]!} \left(\frac{1}{2}\right)^{(N+m+N-m)/2}$$

$$P_N(m) = \frac{N!}{[(N+m)/2]![(N-m)/2]!} \left(\frac{1}{2}\right)^{2N/2} = \frac{N!}{[(N+m)/2]![(N-m)/2]!} \left(\frac{1}{2}\right)^N$$

1.2 THE SIMPLE RANDOM WALK PROBLEM IN ONE-DIMENSION

An Example:

Suppose that $p = q = \frac{1}{2}$ and that $N=3$

The possible numbers of steps to the right are $n_1 = 0, 1, 2, 3$

The corresponding displacements are $m = -3, -1, 1, 3$

The corresponding probabilities are $P_N(m)$

$$P_N(m) = \frac{N!}{[(N+m)/2]![(N-m)/2]!} p^{(N+m)/2} (1-p)^{(N-m)/2}$$

$$P_N(m) = \frac{N!}{[(N+m)/2]![(N-m)/2]!} \left(\frac{1}{2}\right)^N$$

$$P_3(m) = \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8} \quad \text{HOW??}$$

1.2 THE SIMPLE RANDOM WALK PROBLEM IN ONE-DIMENSION

n_1	n_2	m	N	$(N + m)/2$	$(N - m)/2$	$p^{(N+m)/2}$	$(1 - p)^{(N-m)/2}$	$\frac{N!}{[(N + m)/2]! [(N - m)/2]!}$	$P_N(m)$
3	0	3	3	3	0	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^0$	1	$\frac{1}{8}$
2	1	1	3	2	1	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^1$	3	$\frac{3}{8}$
1	2	-1	3	1	2	$\left(\frac{1}{2}\right)^1$	$\left(\frac{1}{2}\right)^2$	3	$\frac{3}{8}$
0	3	-3	3	0	3	$\left(\frac{1}{2}\right)^0$	$\left(\frac{1}{2}\right)^3$	1	$\frac{1}{8}$

PROBABILITY CALCULATIONS

In the simple random walk problem, what are the probabilities that after $N=3$ steps the particle can be found at the locations defined by the displacements $m=-3,-1,1,3$?

PROBABILITY CALCULATIONS

In the simple random walk problem, what are the probabilities that after $N=3$ steps the particle can be found at the locations defined by the displacements $m=-3,-1,1,3$?

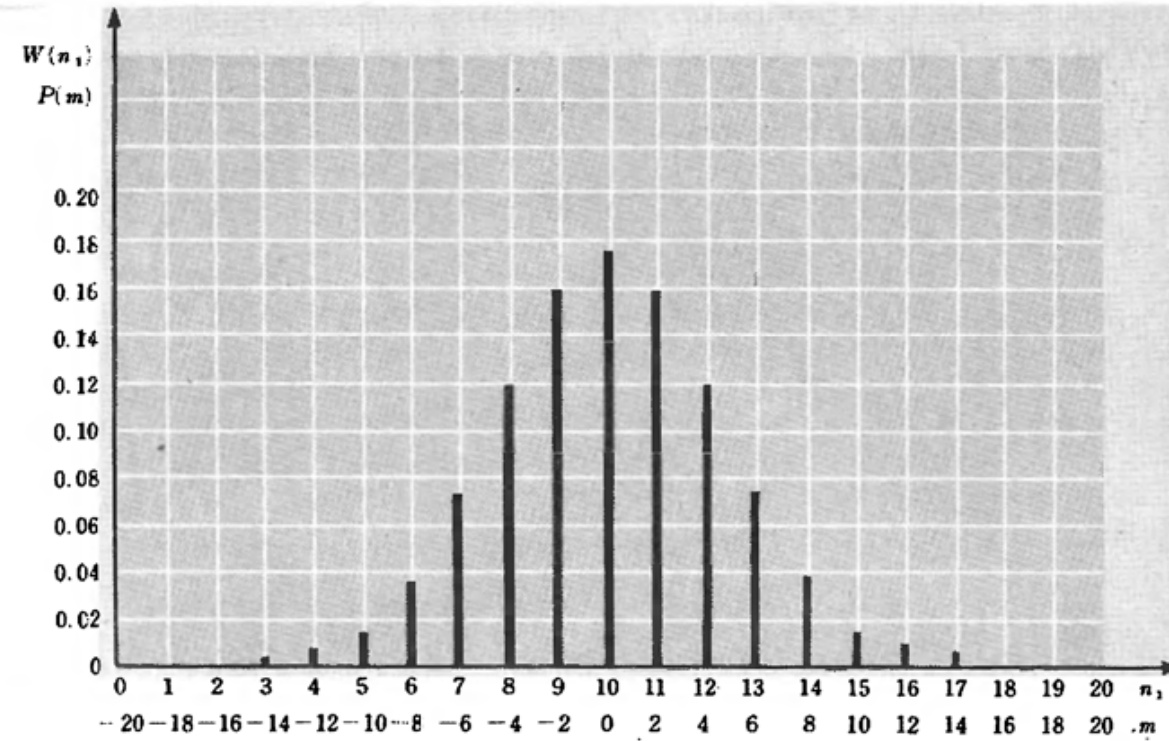
$$P_3(-3) = \frac{3!}{[(3+(-3))/2]![(3-(-3))/2]!} \left(\frac{1}{2}\right)^3 = \frac{3!}{(\frac{0}{2})!(\frac{6}{2})!} \left(\frac{1}{2}\right)^3 = \frac{3!}{0!3!} \left(\frac{1}{2}\right)^3 = (1) \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{1}{8}$$

$$P_3(-1) = \frac{3!}{[(3+(-1))/2]![(3-(-1))/2]!} \left(\frac{1}{2}\right)^3 = \frac{3!}{1!2!} \left(\frac{1}{2}\right)^3 = \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 1} \left(\frac{1}{2}\right)^3 = (3) \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{3}{8}$$

$$P_3(1) = \frac{3!}{[(3+1)/2]![(3-1)/2]!} \left(\frac{1}{2}\right)^3 = \frac{3!}{2!1!} \left(\frac{1}{2}\right)^3 = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} \left(\frac{1}{2}\right)^3 = (3) \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{3}{8}$$

$$P_3(3) = \frac{3!}{[(3+3)/2]![(3-3)/2]!} \left(\frac{1}{2}\right)^3 = \frac{3!}{(\frac{6}{2})!(\frac{0}{2})!} \left(\frac{1}{2}\right)^3 = \frac{3!}{3!0!} \left(\frac{1}{2}\right)^3 = (1) \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{1}{8}$$

1.2 THE SIMPLE RANDOM WALK PROBLEM IN ONE-DIMENSION



After N random steps the probability of the particle being a distance N steps away from the origin is very small, while the probability of it being in the vicinity of the origin is largest.

PROBABILITY CALCULATIONS

In the special case of the simple random walk problem when $p = q = \frac{1}{2}$, what are the probabilities that after $N=4$ steps the particle can be found at the locations defined by the displacements $m=-4, -3, -2, -1, 0, 1, 2, 3, 4$?

**Odd displacement values are not possible!*

Practice Problem

THE SOLUTION

n_1	n_2	m	N	$(N + m)/2$	$(N - m)/2$	$p^{(N+m)/2}$	$(1 - p)^{(N-m)/2}$	$\frac{N!}{[(N + m)/2]! [(N - m)/2]!}$	$P_N(m)$
4	0	4	4	4	0	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^0$	1	$\frac{1}{16}$
3	1	2	4	3	1	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^1$	4	$\frac{4}{16}$
2	2	0	4	2	2	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^2$	6	$\frac{6}{16}$
1	3	-2	4	1	3	$\left(\frac{1}{2}\right)^1$	$\left(\frac{1}{2}\right)^3$	4	$\frac{4}{16}$
0	4	-4	4	0	4	$\left(\frac{1}{2}\right)^0$	$\left(\frac{1}{2}\right)^4$	1	$\frac{1}{16}$

Practice Problem

THE SOLUTION

n_1	n_2	m	N	$(N + m)/2$	$(N - m)/2$	$p^{(N+m)/2}$	$(1 - p)^{(N-m)/2}$	$\frac{N!}{[(N + m)/2]! [(N - m)/2]!}$	$P_N(m)$
4	0	4	4	4	0	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^0$	1	$\frac{1}{16}$
3	1	2	4	3	1	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^1$	4	$\frac{4}{16} = \frac{1}{4}$
2	2	0	4	2	2	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^2$	6	$\frac{6}{16} = \frac{3}{8}$
1	3	-2	4	1	3	$\left(\frac{1}{2}\right)^1$	$\left(\frac{1}{2}\right)^3$	4	$\frac{4}{16} = \frac{1}{4}$
0	4	-4	4	0	4	$\left(\frac{1}{2}\right)^0$	$\left(\frac{1}{2}\right)^4$	1	$\frac{1}{16}$

Practice Problem

In the simple random walk problem, what are the probabilities that after $N=4$ steps the particle can be found at the locations defined by the displacements $m=-4,-2,0,2,4$?

$$P_4(-4) = \frac{4!}{[(4+(-4))/2]![(4-(-4))/2]!} \left(\frac{1}{2}\right)^4 = \frac{4!}{\left(\frac{0}{2}\right)!\left(\frac{8}{2}\right)!} \left(\frac{1}{2}\right)^4 = \frac{4!}{0!4!} \left(\frac{1}{2}\right)^4 = (1) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{1}{16}$$

$$P_4(-2) = \frac{4!}{[(4+(-2))/2]![(4-(-2))/2]!} \left(\frac{1}{2}\right)^4 = \frac{4!}{\left(\frac{2}{2}\right)!\left(\frac{6}{2}\right)!} \left(\frac{1}{2}\right)^4 = \frac{4!}{1!3!} \left(\frac{1}{2}\right)^4 = (4) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{4}{16}$$

$$P_4(0) = \frac{4!}{[(4+0)/2]![(4-0)/2]!} \left(\frac{1}{2}\right)^4 = \frac{4!}{\left(\frac{4}{2}\right)!\left(\frac{4}{2}\right)!} \left(\frac{1}{2}\right)^4 = \frac{4!}{2!2!} \left(\frac{1}{2}\right)^4 = (6) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{6}{16}$$

$$P_4(2) = \frac{4!}{[(4+2)/2]![(4-2)/2]!} \left(\frac{1}{2}\right)^4 = \frac{4!}{\left(\frac{6}{2}\right)!\left(\frac{2}{2}\right)!} \left(\frac{1}{2}\right)^4 = \frac{4!}{3!1!} \left(\frac{1}{2}\right)^4 = (4) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{4}{16}$$

$$P_4(4) = \frac{4!}{[(4+4)/2]![(4-4)/2]!} \left(\frac{1}{2}\right)^4 = \frac{4!}{\left(\frac{8}{2}\right)!\left(\frac{0}{2}\right)!} \left(\frac{1}{2}\right)^4 = \frac{4!}{4!0!} \left(\frac{1}{2}\right)^4 = (1) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{1}{16}$$

Practice Problem

In the simple random walk problem, what are the probabilities that after $N=4$ steps the particle can be found at the locations defined by the displacements $m=-4,-2,0,2,4$?

$$P_4(-4) = \frac{4!}{[(4+(-4))/2]![(4-(-4))/2]!} \left(\frac{1}{2}\right)^4 = \frac{4!}{\binom{0}{2}!\binom{8}{2}!} \left(\frac{1}{2}\right)^4 = \frac{4!}{0!4!} \left(\frac{1}{2}\right)^4 = (1) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{1}{16}$$

$$P_4(-2) = \frac{4!}{[(4+(-2))/2]![(4-(-2))/2]!} \left(\frac{1}{2}\right)^4 = \frac{4!}{\binom{2}{2}!\binom{6}{2}!} \left(\frac{1}{2}\right)^4 = \frac{4!}{1!3!} \left(\frac{1}{2}\right)^4 = (4) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{4}{16} = \frac{1}{4}$$

$$P_4(0) = \frac{4!}{[(4+0)/2]![(4-0)/2]!} \left(\frac{1}{2}\right)^4 = \frac{4!}{\binom{4}{2}!\binom{4}{2}!} \left(\frac{1}{2}\right)^4 = \frac{4!}{2!2!} \left(\frac{1}{2}\right)^4 = (6) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{6}{16} = \frac{3}{8}$$

$$P_4(2) = \frac{4!}{[(4+2)/2]![(4-2)/2]!} \left(\frac{1}{2}\right)^4 = \frac{4!}{\binom{6}{2}!\binom{2}{2}!} \left(\frac{1}{2}\right)^4 = \frac{4!}{3!1!} \left(\frac{1}{2}\right)^4 = (4) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{4}{16} = \frac{1}{4}$$

$$P_4(4) = \frac{4!}{[(4+4)/2]![(4-4)/2]!} \left(\frac{1}{2}\right)^4 = \frac{4!}{\binom{8}{2}!\binom{0}{2}!} \left(\frac{1}{2}\right)^4 = \frac{4!}{4!0!} \left(\frac{1}{2}\right)^4 = (1) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{1}{16}$$

COMMON MISTAKES!!

$$\frac{3!}{1!2!} \left(\frac{1}{2}\right)^3 = \frac{3! \cdot 2! \cdot 1!}{1! \cdot 2! \cdot 1!} \left(\frac{1}{2}\right)^3$$

$$\frac{3!}{[(3 + (-1))!/2]! [(3 - (-1))!/2]!} \left(\frac{1}{2}\right)^3 = \frac{3!}{1!2!} \frac{13}{2}$$

$$\frac{3.2.1}{2.1.1} \left(\frac{1}{2}\right)^3 = \frac{6}{16} \text{ or } \frac{3}{16}$$

$$\mathbf{P}_3(3) = \frac{3!}{[3!+3!/2!][3!-3!/2!]} \left(\frac{1}{2}\right)^3$$

1.3 GENERAL DISCUSSION OF MEAN VALUES

Revisiting our previous-knowledge discussion two-weeks ago:

- The sum of ALL the probabilities is 1

$$\sum_{j=0}^{\infty} P(j) = 1$$

- The average or (mean)?

$$\langle j \rangle = \sum_{j=0}^{\infty} \frac{j N(j)}{N} = \sum_{j=0}^{\infty} j P(j)$$

- In general, the average value of some function of j is given by

$$\langle f(j) \rangle = \sum_{j=0}^{\infty} f(j) P(j)$$

1.3 GENERAL DISCUSSION OF MEAN VALUES

Revisiting our previous-knowledge discussion two-weeks ago:

- The amount of spread in a distribution with respect to the average?

$$\Delta j = j - \langle j \rangle$$

- The variance of the distribution σ^2 ?

$$\sigma^2 = \langle (\Delta j)^2 \rangle = \langle j^2 \rangle - \langle j \rangle^2$$

- The standard deviation of the distribution σ ?

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$$

$$^{**} \langle j^2 \rangle \geq \langle j \rangle^2$$

They are equal when $\sigma = 0$

1.3 GENERAL DISCUSSION OF MEAN VALUES

- Let u be a variable which can assume any of the M discrete values

$$u_1, u_2, u_3, \dots, u_M$$

with respective probabilities $P(u_1), P(u_2), P(u_3), \dots, P(u_M)$

The mean or average value of u is denoted by $\bar{u}$ and is defined by

$$\bar{u} = \langle u \rangle = \frac{u_1 P(u_1) + u_2 P(u_2) + \dots + u_M P(u_M)}{P(u_1) + P(u_2) + \dots + P(u_M)} = \frac{\sum_{i=1}^M u_i P(u_i)}{\sum_{i=1}^M P(u_i)} = \frac{\sum_{i=1}^M u_i P(u_i)}{1} \quad (1)$$

$$\text{Since } P(u_1) + P(u_2) + \dots + P(u_M) = \sum_{i=1}^M P(u_i) = 1$$

This is the so called “Normalization Condition”

- If all the values of the variable u are equally probable, then the mean value of the distribution will be

$$\bar{u} = \frac{u_1 + u_2 + \dots + u_M}{M} = \frac{\sum_{i=1}^M u_i}{M}$$

1.3 GENERAL DISCUSSION OF MEAN VALUES

- In general,

$$\overline{f(u)} = \langle f(u) \rangle = \sum_{i=1}^M f(u_i) P(u_i)$$

- Note the following simple properties,

1.
$$\overline{f(u) + g(u)} = \sum_{i=1}^M [f(u_i) + g(u_i)] P(u_i) =$$

$$\sum_{i=1}^M f(u_i) P(u_i) + \sum_{i=1}^M g(u_i) P(u_i) = \overline{f(u)} + \overline{g(u)} \quad (\text{Additivity})$$

2.
$$\overline{f(u) - g(u)} = \sum_{i=1}^M [f(u_i) - g(u_i)] P(u_i) =$$

$$\sum_{i=1}^M f(u_i) P(u_i) - \sum_{i=1}^M g(u_i) P(u_i) = \overline{f(u)} - \overline{g(u)} \quad (\text{Linearity})$$

3.
$$\overline{cf(u)} = c \overline{f(u)} \quad (\text{Scaling})$$

4. If u and v are statically independent, then $\overline{f(u) \cdot g(u)} = \overline{f(u)} \cdot \overline{g(u)}$ (Independence)

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

First, let's VERIFY the normalization

$$\sum_{n_1=0}^N W_N(n_1) = 1$$

Using the Binomial theorem & your current knowledge since $p + q = 1$,

$$\sum_{n_1=0}^N W(n_1) = \sum_{n_1=0}^N \frac{N!}{n_1! (N - n_1)!} p^{n_1} q^{N-n_1} = (p + q)^N = 1^N = 1$$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Second, let's CALCULATE $\overline{n_1}$ the mean number of steps to the right

$$\overline{n_1} = \sum_{n_1=0}^N n_1 W(n_1) = \sum_{n_1=0}^N \frac{N!}{n_1! (N - n_1)!} p^{n_1} q^{N-n_1} n_1$$

$$n_1 p^{n_1} = p \frac{\delta}{\delta p} (p^{n_1})$$

$$\begin{aligned} \overline{n_1} &= \sum_{n_1=0}^N \frac{N!}{n_1! (N - n_1)!} p^{n_1} q^{N-n_1} n_1 = \sum_{n_1=0}^N \frac{N!}{n_1! (N - n_1)!} \left[p \frac{\delta}{\delta p} (p^{n_1}) \right] q^{N-n_1} \\ &= p \frac{\delta}{\delta p} \left[\sum_{n_1=0}^N \frac{N!}{n_1! (N - n_1)!} (p^{n_1}) q^{N-n_1} \right] = p \frac{\delta}{\delta p} (p + q)^N = pN(p + q)^{N-1} \\ &= pN(1)^{N-1} = pN \end{aligned}$$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Exercise

CALCULATE $\overline{n_2}$ the mean number of
steps to the left then find $(\overline{n_1} + \overline{n_2})$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Exercise

CALCULATE $\overline{n_2}$ the mean number of steps to the left then find $(\overline{n_1} + \overline{n_2})$

$$\overline{n_2} = \sum_{n_2=0}^N n_2 W(n_2) = \sum_{n_2=0}^N \frac{N!}{n_2! (N - n_2)!} p^{N-n_2} q^{n_2} n_2$$

$$\begin{aligned} \overline{n_2} &= \sum_{n_2=0}^N \frac{N!}{n_2! (N - n_2)!} p^{N-n_2} q^{n_2} n_2 = \sum_{n_2=0}^N \frac{N!}{n_2! (N - n_2)!} \left[q \frac{\delta}{\delta q} (q^{n_2}) \right] p^{N-n_2} \\ &= q \frac{\delta}{\delta q} \left[\sum_{n_2=0}^N \frac{N!}{n_2! (N - n_2)!} (q^{n_2}) p^{N-n_2} \right] = q \frac{\delta}{\delta q} (p + q)^N = q N (p + q)^{N-1} \\ &= q N (1)^{N-1} = qN \end{aligned}$$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Exercise

- Find $(\overline{n_1} + \overline{n_2})$

$$\overline{n_1} + \overline{n_2} = N(p + q) = N(1) = N = \overline{n_1 + n_2}$$

- Find the mean displacement $\overline{m}$

$$\overline{m} = \overline{n_1 - n_2} = \overline{n_1} - \overline{n_2} = N(p - q)$$

In the case of complete symmetry between right & left directions $p=q$

we find that $\overline{m} = 0$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

- Calculation of the dispersion $\overline{(\Delta n_1)^2}$:

$$\overline{(\Delta n_1)^2} = \overline{(n_1 - \bar{n}_1)^2} = \overline{n_1^2} - \bar{n}_1^2$$

We know already that

$$\bar{n}_1 = Np$$

Computing $\overline{n_1^2}$,

$$\overline{n_1^2} = \sum_{n_1=0}^N n_1^2 W(n_1) = \sum_{n_1=0}^N \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1} n_1^2$$

$$n_1^2 p^{n_1} = n_1 \left(p \frac{\delta}{\delta p} \right) (p^{n_1}) = \left(p \frac{\delta}{\delta p} \right)^2 (p^{n_1})$$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

$$\begin{aligned}
 \overline{n_1^2} &= \sum_{n_1=0}^N n_1^2 W(n_1) = \sum_{n_1=0}^N \frac{N!}{n_1!(N-n_1)!} \left(p \frac{\delta}{\delta p}\right)^2 (p^{n_1}) q^{N-n_1} \\
 &= \left(p \frac{\delta}{\delta p}\right)^2 \sum_{n_1=0}^N \frac{N!}{n_1!(N-n_1)!} (p^{n_1}) q^{N-n_1} \\
 &= \left(p \frac{\delta}{\delta p}\right)^2 (p+q)^N \\
 &= \left(p \frac{\delta}{\delta p}\right) [pN(p+q)^{N-1}] \\
 &= p[N(p+q)^{N-1} + pN(N-1)(p+q)^{N-2}] \quad , p+q=1 \\
 &= p[N + pN(N-1)] \\
 &= Np[1 + pN - p] \\
 &= (Np)^2 + Npq \\
 \overline{n_1^2} &= \bar{n}_1^2 + Npq
 \end{aligned}$$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

$$\overline{n_1^2} = \bar{n}_1^2 + Npq \quad \longrightarrow \quad \overline{n_1^2} - \bar{n}_1^2 = Npq$$

Back to the Calculation of the dispersion $\overline{(\Delta n_1)^2}$

$$\overline{(\Delta n_1)^2} = \overline{(n_1 - \bar{n}_1)^2} = \overline{n_1^2} - \bar{n}_1^2$$

$$\overline{(\Delta n_1)^2} = Npq$$

Calculating the root mean square deviation

$$\Delta^* n_1 = \sqrt{\overline{(\Delta n_1)^2}}$$

$$\Delta^* n_1 = \sqrt{\overline{(\Delta n_1)^2}} = \sqrt{Npq}$$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

The relative width of the distribution is

$$\frac{\Delta^* n_1}{\bar{n}_1} = \frac{\sqrt{Npq}}{Np} = \sqrt{\frac{q}{p}} \frac{1}{\sqrt{N}}$$

For the special case of $p = q = \frac{1}{2}$

$$\frac{\Delta^* n_1}{\bar{n}_1} = \frac{\sqrt{Npq}}{Np} = \sqrt{\frac{q}{p}} \frac{1}{\sqrt{N}} = \sqrt{\frac{1/2}{1/2}} \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}}$$

As N increases, the mean value $\bar{n}_1$ increases like N but the width $\Delta^* n_1$ increases like $\sqrt{N}$. Hence, the relative width $\frac{\Delta^* n_1}{\bar{n}_1}$ decreases with increasing N like $\frac{1}{\sqrt{N}}$.

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

- The dispersion of the displacement m :

$$m = n_1 - n_2 = n_1 - (N - n_1) = 2n_1 - N$$

$$\Delta m = m - \bar{m} = [2n_1 - N] - [2\bar{n}_1 - N]$$

$$\Delta m = 2(n_1 - \bar{n}_1)$$

$$\Delta m = 2\Delta n_1$$

$$(\Delta m)^2 = 4(\Delta n_1)^2$$

Taking averages of both sides, and using the dispersion $\overline{(\Delta n_1)^2}$ calculation $\overline{(\Delta n_1)^2} = Npq$

$$\overline{(\Delta m)^2} = 4\overline{(\Delta n_1)^2} = 4Npq$$

For the special case of $p = q = \frac{1}{2}$

$$\overline{(\Delta m)^2} = 4\overline{(\Delta n_1)^2} = 4N \frac{1}{2} \frac{1}{2} = N$$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

In the random walk problem, if the total number of steps is N .

- Discuss the binomial distribution function and its significance
- Find the probability as a function of the displacement $P_N(m)$
- Verify the normalization condition $\sum_{n_1=0}^N W_N(n_1) = 1$
- Show that the mean number of steps to the right is $\bar{n}_1 = Np$
- Show that the mean number of steps to the left is $\bar{n}_2 = Nq$
- Find $\bar{n}_1 + \bar{n}_2$
- Find the mean of the displacement $\bar{m}$
- In the case of complete symmetry where $p=q$, show that $\bar{m} = 0$
- Evaluate the dispersion $\overline{(\Delta n_1)^2}$
- root mean square deviation $\Delta^* n_1$
- Evaluate the relative width of the distribution $\frac{\Delta^* n_1}{\bar{n}_1}$
- In the case of complete symmetry where $p=q$, evaluate $\frac{\Delta^* n_1}{\bar{n}_1}$
- Evaluate the dispersion of the displacement $\overline{(\Delta m)^2}$
- In the case of complete symmetry where $p=q$, evaluate $\overline{(\Delta m)^2}$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

		$p \neq q$	$p = q$
Probability in (N, n_1)	$W_N(n_1)$	$\frac{N!}{n_1!n_2!} p^{n_1} q^{n_2}$	$\frac{N!}{n_1! (N - n_1)!} \left(\frac{1}{2}\right)^N$
Probability in (N, m)	$P_N(m)$	$\frac{N!}{[(N + m)/2]! [(N - m)/2]!} p^{(N+m)/2} q^{(N-m)/2}$	$\frac{N!}{[(N + m)/2]! [(N - m)/2]!} \left(\frac{1}{2}\right)^N$
Mean of n_1 & n_2	$\bar{n}_1 + \bar{n}_2$	$Np + Nq = N(p + q)$	N
Dispersion of n_1	$\overline{(\Delta n_1)^2}$	Npq	$N/4$
Root mean square deviation of n_1	$\Delta^* n_1$	$\sqrt{\overline{(\Delta n_1)^2}} = \sqrt{Npq}$	$\sqrt{N/4}$
Mean of m	$\bar{m}$	$\bar{n}_1 - \bar{n}_2 = N(p - q)$	0
Dispersion of m	$\overline{(\Delta m)^2}$	$4Npq$	N
Root mean square deviation of m	$\Delta^* m$	$\sqrt{\overline{(\Delta m)^2}} = \sqrt{4Npq}$	$\sqrt{N}$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Example:

Consider the case of $N = 100$ steps, where $p = q = \frac{1}{2}$. Then the mean number of steps to the right $\overline{n_1}$ (or to the left) is 50; the mean displacement is $\overline{m} = 0$. The root mean square displacement is?

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Example:

Consider the case of $N = 100$ steps, where $p = q = \frac{1}{2}$. Then the mean number of steps to the right $\overline{n_1}$ (or to the left) is 50; the mean displacement is $\overline{m} = 0$. The root mean square displacement is?

$$\overline{(\Delta m)^2} = N$$
$$\Delta^* m = \sqrt{\overline{(\Delta m)^2}} = \sqrt{N} = \sqrt{100} = 10 \text{ steps}$$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Example:

In the special case of the simple random walk problem when $p = q$, what is the probability that after $N=7$ steps the particle can be found at the location defined by the displacements $m=-3$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Example:

In the special case of the simple random walk problem when $p = q$, what is the probability that after $N=7$ steps the particle can be found at the location defined by the displacements $m=-3$

$$P_N(m) = \frac{N!}{[(N+m)/2]![(N-m)/2]!} \left(\frac{1}{2}\right)^N$$

$$P_7(-3) = \frac{7!}{[(7+(-3))/2]![(7-(-3))/2]!} \left(\frac{1}{2}\right)^7 = \frac{7!}{\left(\frac{4}{2}\right)!\left(\frac{10}{2}\right)!} \left(\frac{1}{2}\right)^7 = \frac{7!}{2!5!} \left(\frac{1}{2}\right)^7 = (21) \frac{1}{128} = \frac{21}{128}$$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Prove that for a variable u the mean value of deviation from the mean is zero.

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Prove that for a variable u the mean value of deviation from the mean is zero.

- **Define the mean:** Let the random variable be u with a mean expected value $\bar{u} = \sum_{i=1}^N (u_i) / N$
- **Define the deviation from the mean:** The deviation of a single data point u_i from the mean $\bar{u}$ is $(u_i - \bar{u})$.
- **Define the sum of all deviations:** $\sum_{i=1}^N (u_i - \bar{u})$
- **Expand the sum:** $\sum_{i=1}^N (u_i - \bar{u}) = \sum_{i=1}^N (u_i) - \sum_{i=1}^N (\bar{u})$
- **Simplify both terms:** $\sum_{i=1}^N (u_i - \bar{u}) = \sum_{i=1}^N (u_i) - N\bar{u} = N\bar{u} - N\bar{u} = 0$
- Therefore, positive and negative deviations from the mean cancel, so the average deviation is zero.

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

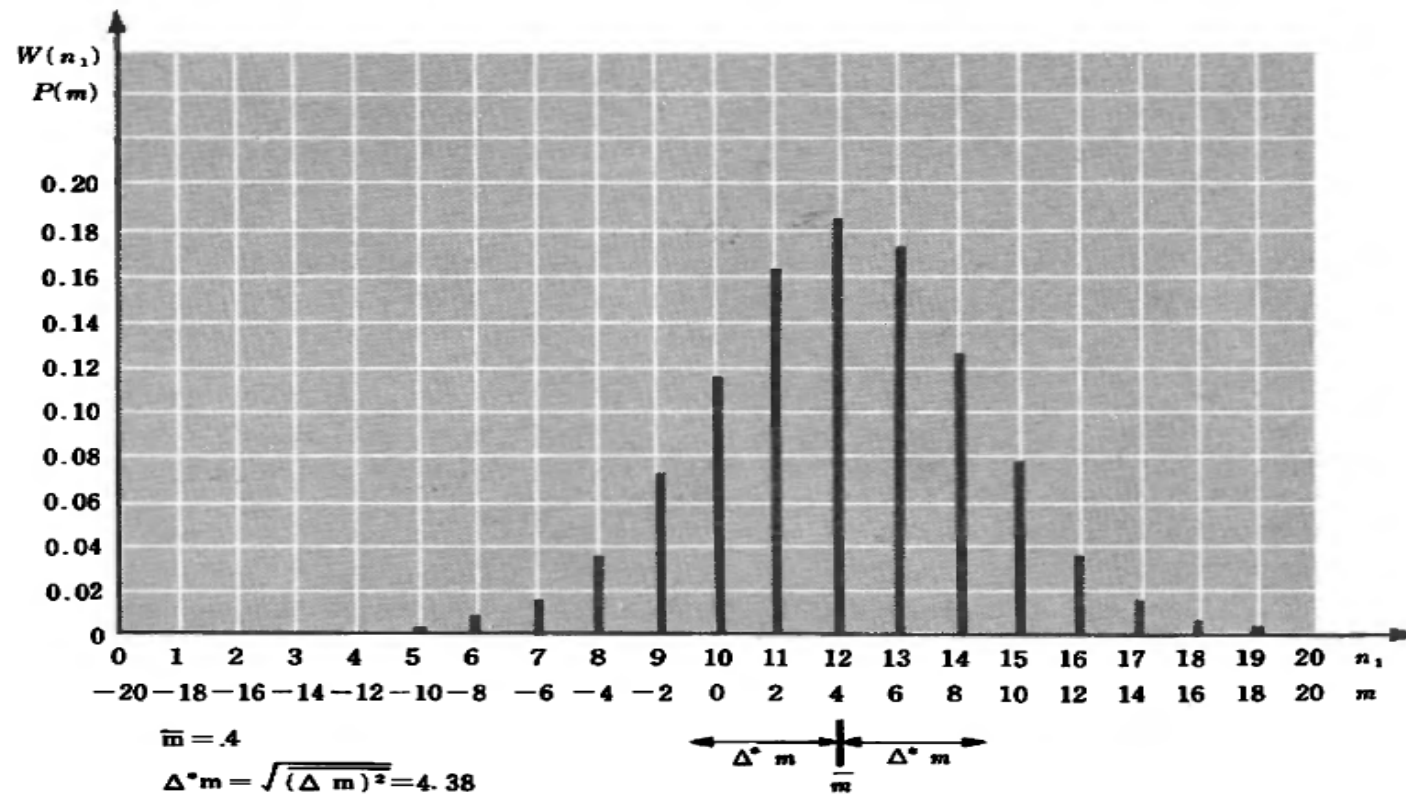
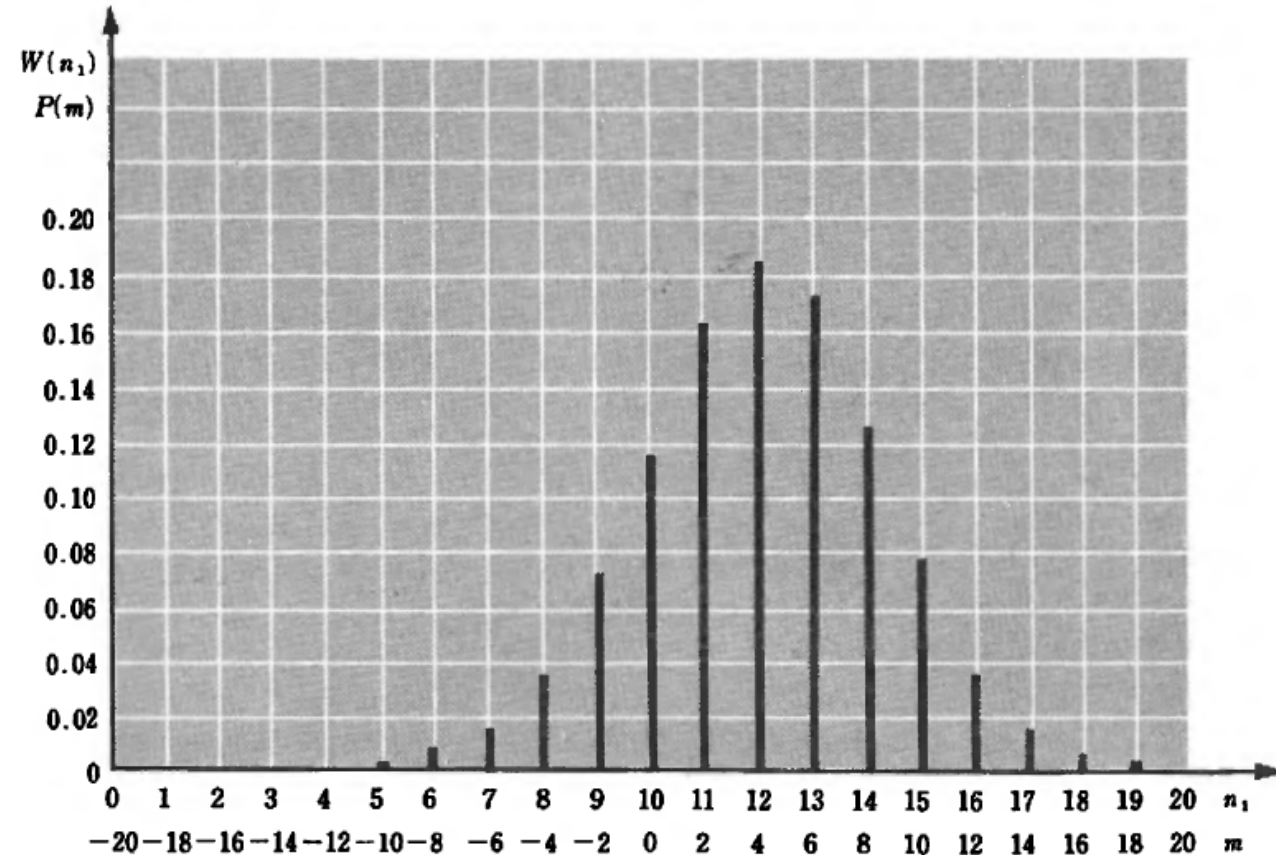


Fig. 1·4·1 Binomial probability distribution for $p = 0.6$ and $q = 0.4$, when $N = 20$ steps. The graph shows again the probability $W(n_1)$ of n_1 right steps, or equivalently, the probability $P(m)$ of a net displacement of m units to the right. The mean values $\bar{m}$ and $(\Delta m)^2$ are also indicated.

Practice Problem

The binomial probability distribution for $p = 0.6$ and $q = 0.4$ when the total number of steps $N = 20$ is shown, calculate:

1. The probability $W(n_1)$ for $n_1 = 12$ & $n_1 = 6$
2. The mean of the displacement $\bar{m}$
3. The root mean square deviation in Δ^*m
4. The relative quantity $\Delta^*m / \bar{m}$



SOLUTION

1. $p \neq q$, $W_N(n_1) = W(n_1) = \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1}$
 $W_{20}(12) = \frac{20!}{12!(20-12)!} (0.6)^{12} (0.4)^{(20-12)} = \frac{20!}{12! 8!} (0.6)^{12} (0.4)^8 = 0.18291$
 $W_{20}(6) = \frac{20!}{6!(20-6)!} (0.6)^6 (0.4)^{(20-6)} = \frac{20!}{6! 14!} (0.6)^6 (0.4)^{14} = 0.00485$
2. The mean $\bar{m}$: $\bar{m} = \overline{n_1 - n_2} = N(p - q) = 20(0.6 - 0.4) = 4$
3. The root mean square deviation $\Delta^* m$:

$$\Delta^* m = \sqrt{(\Delta m)^2} = \sqrt{4Npq} = \sqrt{4(20)(0.6)(0.4)} = \sqrt{19.2} = 4.38$$
4. The relative width $\frac{\Delta^* m}{\bar{m}}$: $\frac{\Delta^* m}{\bar{m}} = \frac{\sqrt{(\Delta m)^2}}{\bar{m}} = \frac{4.38}{4} = 1.095$

Practice Problem

The binomial probability distribution for $p = 0.7$ and $q = 0.3$ when the total number of steps $N = 20$ is shown, calculate:

1. The probability $W(n_1)$ for $n_1 = 14$ & $n_1 = 10$
2. The mean of the displacement $\bar{m}$
3. The root mean square deviation in $\Delta^* m$
4. The relative quantity $\Delta^* m / \bar{m}$

Practice Problem

The binomial probability distribution for $p = 0.7$ and $q = 0.3$ when the total number of steps $N = 20$ is shown, calculate:

1. The probability $W(n_1)$ for $n_1 = 14$ & $n_1 = 10$

$$p \neq q, \quad W_N(n_1) = W(n_1) = \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1}$$

$$W_{20}(14) = 0.1916, \quad W_{20}(10) = 0.308$$

2. The mean of the displacement $\bar{m}$

$$\bar{m} = \overline{n_1 - n_2} = N(p - q) = 20(0.7 - 0.3) = 8$$

3. The root mean square deviation in $\Delta^* m$

$$\Delta^* m = \sqrt{(\Delta m)^2} = \sqrt{4Npq} = \sqrt{4(20)(0.7)(0.3)} = 4.098$$

4. The relative quantity $\Delta^* m / \bar{m}$

$$\frac{\Delta^* m}{\bar{m}} = \frac{\sqrt{(\Delta m)^2}}{\bar{m}} = \frac{4.098}{8} = 0.5123$$

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Previously Solved

In the case of the simple random walk problem where $N=4$, suppose that $p = q$.

- a) What are all the possible values of the displacement m .
- b) Calculate the probabilities of being located at each value of m in (a).
- c) What is the sum of probabilities calculated in (b).

1.4 CALCULATION OF MEAN VALUES FOR THE RANDOM WALK PROBLEM

Previously Solved

In the case of the simple random walk problem where $N=4$, suppose that $p = q$.

a) What are all the possible values of the displacement m .

n_1	4	3	2	1	0
n_2	0	1	2	3	4
$m = n_1 - n_2$	4	2	0	-2	-4

a) Calculate the probabilities of being located at each value of m in (a).

$$P_4(-4) = \frac{1}{16}, P_4(-2) = \frac{4}{16}, P_4(0) = \frac{6}{16}, P_4(2) = \frac{4}{16}, P_4(4) = \frac{1}{16}$$

a) What is the sum of probabilities calculated in (b).

$$P_4(-4) + P_4(-2) + P_4(0) + P_4(2) + P_4(4) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = 1$$

1.7 PROBABILITY DISTRIBUTIONS INVOLVING SEVERAL VARIABLES

As a generalization of the previous results, we will consider two variables $\mathbf{u}$ and $\mathbf{v}$ having different values u_i and v_j where, $i = 1, 2, \dots, M$ and $j = 1, 2, \dots, N$

Let $P(u_i, v_j)$ be the probability of occurrence of the event $u = u_i$ and $v = v_j$
then the normalization condition applies

$$\sum_{i=1}^M \sum_{j=1}^N P(u_i, v_j) = 1$$

If we divide the probability $P(u_i, v_j)$ into two sections;

- 1) $P_v(u_i)$ which depends only on the variable u .
- 2) $P_u(v_j)$ which depends only on the variable v .

Then the normalization condition is still applicable

$$\sum_{i=1}^M P_v(u_i) = \sum_{i=1}^M P(u_i, v_j) = 1 \quad \text{and} \quad \sum_{j=1}^N P_u(v_j) = \sum_{j=1}^N P(u_i, v_j) = 1$$

In the special case, where the variables are statically independent (uncorrelated), the value of $P(u_i, v_j)$ can be written as

$$P(u_i, v_j) = P_u(u_i) P_v(v_j)$$

1.7 PROBABILITY DISTRIBUTIONS INVOLVING SEVERAL VARIABLES

For two functions $F(u)$ and $G(v)$ show that "the average of the product is equal to the product of the average if and only if the variables are statistically independent".

$$\overline{F(u)} = \sum_{i=1}^M \sum_{j=1}^N P(u_i, v_j) F(u_i) \neq \sum_{i=1}^M P_v(u_i) F(u_i)$$

$$\overline{G(v)} = \sum_{i=1}^M \sum_{j=1}^N P(u_i, v_j) G(v_j) = \sum_{j=1}^N P_u(v_j) G(v_j)$$

$$\overline{F(u)G(v)} = \sum_{i=1}^M \sum_{j=1}^N P(u_i, v_j) F(u_i) P(u_i, v_j) G(v_j) = \sum_{i=1}^M P_v(u_i) F(u_i) \sum_{j=1}^N P_u(v_j) G(v_j)$$

$$\overline{F(u)G(v)} = \overline{F(u)} \overline{G(v)}$$

1.8 COMMENTS ON CONTINUOUS PROBABILITY DISTRIBUTIONS

If the two variables **u** and **v** are continuous variables, then we use the continuous analog of the summation which is the integral.

Discrete Variables	Continuous Variables
$\sum_{i=1}^M P(u_i) = 1$	$\int_0^{\infty} P(u) du = 1$
$\overline{F(u)} = \sum_{i=1}^M P(u_i) F(u_i)$	$\overline{F(u)} = \int_0^{\infty} F(u) P(u) du$
$\sum_{i=1}^M \sum_{j=1}^N P(u_i, v_j) = 1$	$\int_0^{\infty} \int_0^{\infty} P(u, v) du dv = 1$
$\overline{F(u, v)} = \sum_{i=1}^M \sum_{j=1}^N P(u_i, v_j) F(u_i, v_j)$	$\overline{F(u, v)} = \int_0^{\infty} \int_0^{\infty} P(u, v) F(u, v) du dv$