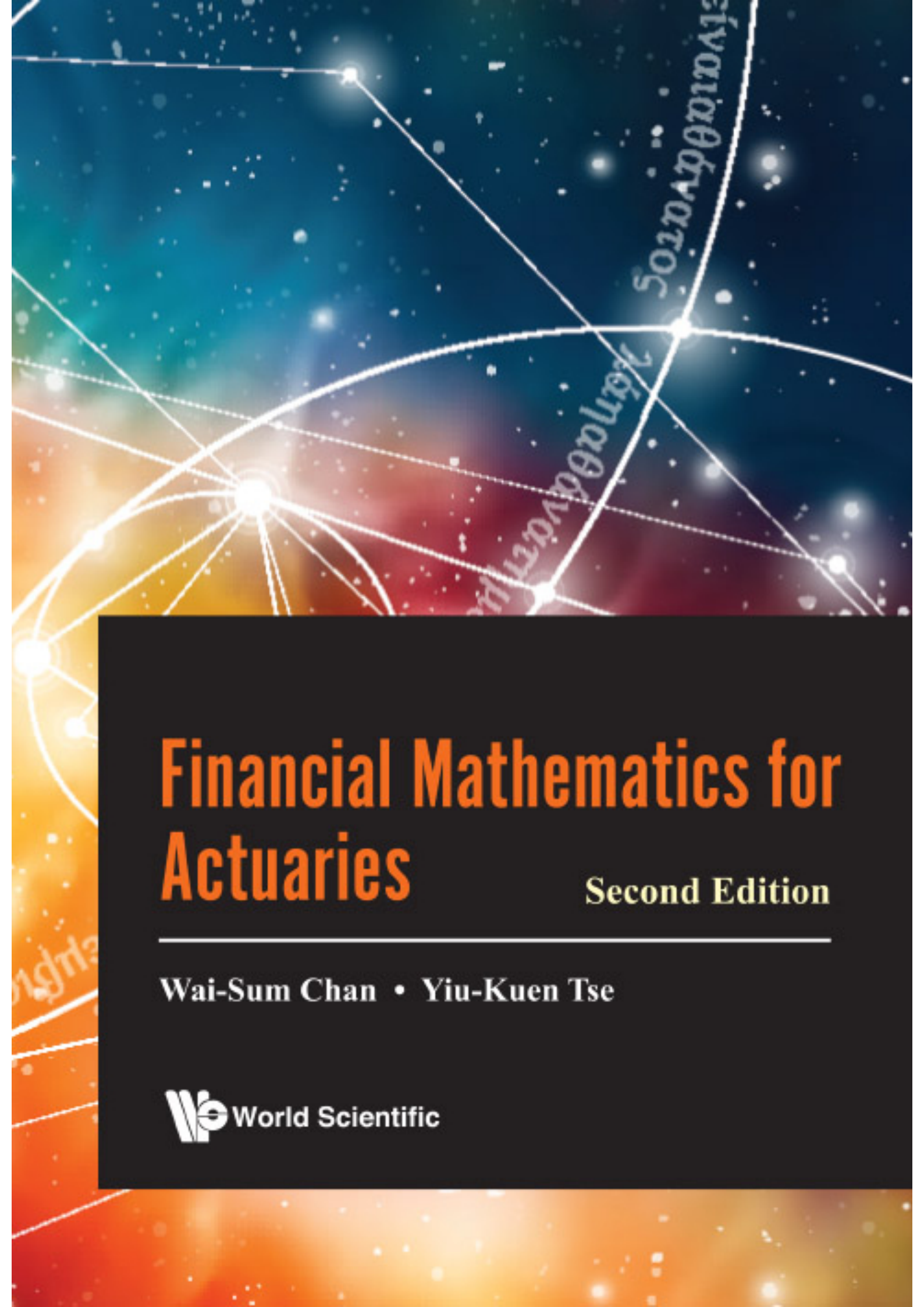




Financial Mathematics for Actuaries

Second Edition

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Interest Accumulation and Time Value of Money

From time to time we are faced with problems of making financial decisions. These may involve anything from borrowing a loan from a bank to purchase a house or a car; or investing money in bonds, stocks or other securities. To a large extent, intelligent wealth management means borrowing and investing wisely.

Financial decision making should take into account the **time value of money**. It is not difficult to see that a dollar received today is worth more than a dollar received one year later. The time value of money depends critically on how interest is calculated. For example, the frequency at which the interest is compounded may be an important factor in determining the cost of a loan.

In this chapter, we discuss the basic principles in the calculation of interest, including the simple- and compound-interest methods, the frequencies of compounding, the effective rate of interest and rate of discount, and the present and future values of a single payment.

Learning Objectives

- *Basic principles in calculation of interest accumulation*
- *Simple and compound interest*
- *Frequency of compounding*
- *Effective rate of interest*
- *Rate of discount*
- *Present and future values of a single payment*

1.1 Accumulation Function and Amount Function

Many financial transactions involve lending and borrowing. The sum of money borrowed is called the **principal**. To compensate the lender for the loss of use of the principal during the loan period the borrower pays the lender an amount of **interest**. At the end of the loan period the borrower pays the lender the **accumulated amount**, which is equal to the sum of the principal plus interest.

We denote $A(t)$ as the accumulated amount at time t , called the **amount function**. Hence, $A(0)$ is the initial principal and

$$I(t) = A(t) - A(t - 1) \quad (1.1)$$

is the interest incurred from time $t - 1$ to time t , namely, in the t th period. For the special case of an initial principal of 1 unit, we denote the accumulated amount at time t by $a(t)$, which is called the **accumulation function**. Thus, if the initial principal is $A(0) = k$, then

$$A(t) = k \times a(t).$$

This assumes that the same accumulation function is used for the amount function irrespective of the initial principal.

1.2 Simple and Compound Interest

Equation (1.1) shows that the growth of the accumulated amount depends on the way the interest is calculated, and vice versa. While theoretically there are

numerous ways of calculating the interest, there are two methods which are commonly used in practice. These are the **simple-interest method** and the **compound-interest method**.

For the simple-interest method, the interest earned over a period of time is proportional to the length of the period. Thus the interest incurred from time 0 to time t , for a principal of 1 unit, is $r \times t$, where r is the constant of proportion called the **rate of interest**. Hence the accumulation function for the simple-interest method is

$$a(t) = 1 + rt, \quad \text{for } t \geq 0, \quad (1.2)$$

and

$$A(t) = A(0)a(t) = A(0)(1 + rt), \quad \text{for } t \geq 0. \quad (1.3)$$

In general the rate of interest may be quoted for any period of time (such as a month or a year). In practice, however, the most commonly used base is the year, in which case the term **annual rate of interest** is used. In what follows we shall maintain this assumption, unless stated otherwise.

Example 1.1: A person borrows \$2,000 for 3 years at simple interest. The rate of interest is 8% per annum. What are the interest charges for year 1 and 2? What is the accumulated amount at the end of year 3?

Solution: The interest charges for year 1 and 2 are both equal to

$$2,000 \times 0.08 = \$160.$$

The accumulated amount at the end of year 3 is

$$2,000 (1 + 0.08 \times 3) = \$2,480.$$



For the compound-interest method, the accumulated amount over a period of time is the principal for the next period. Thus, a principal of 1 unit accumulates to $1 + r$ units at the end of the year, which becomes the principal for the second year. Continuing this process, the accumulation function becomes

$$a(t) = (1 + r)^t, \quad \text{for } t = 0, 1, 2, \dots, \quad (1.4)$$

and the amount function is

$$A(t) = A(0)a(t) = A(0)(1 + r)^t, \quad \text{for } t = 0, 1, 2, \dots. \quad (1.5)$$

Two remarks are noted. First, for the compound-interest method the accumulated amount at the end of a year becomes the principal for the following year.

This is in contrast to the simple-interest method, for which the principal remains unchanged through time. Second, while (1.2) and (1.3) apply for $t \geq 0$, (1.4) and (1.5) hold only for integral $t \geq 0$. As we shall see below, there are alternative ways to define the accumulation function for the compound-interest method when t is not an integer.

Example 1.2: Solve the problem in Example 1.1 using the compound-interest method.

Solution: The interest for year 1 is

$$2,000 \times 0.08 = \$160.$$

For year 2 the principal is

$$2,000 + 160 = \$2,160,$$

so that the interest for the year is

$$2,160 \times 0.08 = \$172.80.$$

The accumulated amount at the end of year 3 is

$$2,000 (1 + 0.08)^3 = \$2,519.42.$$



Compounding has the effect of generating a larger accumulated amount. The effect is especially significant when the rate of interest is high. Table 1.1 shows two samples of the accumulated amounts under simple- and compound-interest methods. It can be seen that when the interest rate is high, compounding the interest induces the principal to grow much faster than the simple-interest method.

With compound interest at 10%, it takes less than 8 years to double the investment. With simple interest at the same rate it takes 10 years to get the same result. Over a 20-year period, an investment with compound interest at 10% will grow 6.73 times. Over a 50-year period, the principal will grow by a phenomenal 117.39 times. When the interest rate is higher the effect of compounding will be even more dramatic.

Table 1.1: Accumulated amount for a principal of \$100

Year	5% interest		10% interest	
	Simple interest (\$)	Compound interest (\$)	Simple interest (\$)	Compound interest (\$)
1	105.00	105.00	110.00	110.00
2	110.00	110.25	120.00	121.00
3	115.00	115.76	130.00	133.10
4	120.00	121.55	140.00	146.41
5	125.00	127.63	150.00	161.05
6	130.00	134.01	160.00	177.16
7	135.00	140.71	170.00	194.87
8	140.00	147.75	180.00	214.36
9	145.00	155.13	190.00	235.79
10	150.00	162.89	200.00	259.37

1.3 Frequency of Compounding

Although the rate of interest is often quoted in annual term, the interest accrued to an investment is often paid more frequently than once a year. For example, a savings account may pay interest at 3% per year, where the interest is credited monthly. In this case, 3% is called the **nominal rate of interest payable 12 times a year**. As we shall see, the **frequency of interest payment** (also called the **frequency of compounding**) makes an important difference to the accumulated amount and the total interest earned. Thus, it is important to define the rate of interest accurately.

To emphasize the importance of the frequency of compounding we use $r^{(m)}$ to denote the **nominal rate of interest payable m times a year**. Thus, m is the frequency of compounding per year and $\frac{1}{m}$ year is the **compounding period** or **conversion period**.

Let t (in years) be an integer multiple of $\frac{1}{m}$, i.e., tm is an integer representing the number of interest-conversion periods over t years. The interest earned over the next $\frac{1}{m}$ year, from time t to $t + \frac{1}{m}$, is

$$a(t) \times r^{(m)} \times \frac{1}{m} = \frac{a(t)r^{(m)}}{m}, \quad \text{for } t = 0, \frac{1}{m}, \frac{2}{m}, \dots$$

Thus, the accumulated amount at time $t + \frac{1}{m}$ is

$$a\left(t + \frac{1}{m}\right) = a(t) + \frac{a(t)r^{(m)}}{m} = a(t) \left[1 + \frac{r^{(m)}}{m}\right], \quad \text{for } t = 0, \frac{1}{m}, \frac{2}{m}, \dots.$$

By recursive substitution, we conclude

$$a(t) = \left[1 + \frac{r^{(m)}}{m}\right]^{mt}, \quad \text{for } t = 0, \frac{1}{m}, \frac{2}{m}, \dots, \quad (1.6)$$

and hence

$$A(t) = A(0) \left[1 + \frac{r^{(m)}}{m}\right]^{mt}, \quad \text{for } t = 0, \frac{1}{m}, \frac{2}{m}, \dots. \quad (1.7)$$

Example 1.3: A person deposits \$1,000 into a savings account that earns 3% interest payable monthly. How much interest will be credited in the first month? What is the accumulated amount at the end of the first month?

Solution: The rate of interest over one month is

$$0.03 \times \frac{1}{12} = 0.25\%,$$

so that the interest earned over one month is

$$1,000 \times 0.0025 = \$2.50,$$

and the accumulated amount after one month is

$$1,000 + 2.50 = \$1,002.50.$$

■

Example 1.4: \$1,000 is deposited into a savings account that pays 3% interest with monthly compounding. What is the accumulated amount after two and a half years? What is the amount of interest earned over this period?

Solution: The investment interval is 30 months. Thus, using (1.7), the accumulated amount is

$$1,000 \left[1 + \frac{0.03}{12}\right]^{30} = \$1,077.78.$$

The amount of interest earned over this period is

$$1,077.78 - 1,000 = \$77.78.$$

■

Example 1.5: Solve the problem in Example 1.4, assuming that the interest is paid quarterly.

Solution: The investment interval is now 10 quarters. With $m = 4$, the accumulated amount is

$$1,000 \left[1 + \frac{0.03}{4} \right]^{10} = \$1,077.58,$$

and the amount of interest earned is \$77.58.



When the loan period is not an integer multiple of the compounding period (i.e., tm is not an integer), care must be taken to define the way interest is calculated over the fraction of the compounding period. Two methods may be considered. First, we may extend (1.6) and (1.7) to apply to any $tm \geq 0$ (not necessarily an integer). Second, we may compute the accumulated value over the largest integral interest-conversion period using (1.7) and then apply the simple-interest method to the remaining fraction of the conversion period. The example below illustrates these two methods.

Example 1.6: What is the accumulated amount for a principal of \$100 after 25 months if the nominal rate of interest is 4% compounded quarterly?

Solution: The accumulation period is $\frac{25}{3} = 8.33$ quarters. Using the first method, the accumulated amount is

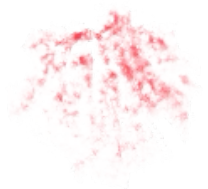
$$100 \left[1 + \frac{0.04}{4} \right]^{8.33} = \$108.64.$$

Using the second method the accumulated amount after 24 months (8 quarters) is

$$100 \left[1 + \frac{0.04}{4} \right]^8 = \$108.29,$$

so that the accumulated amount after 25 months is

$$108.29 \left[1 + 0.04 \times \frac{1}{12} \right] = 108.65.$$



It can be shown that the second method provides a larger accumulation function for any non-integer $tm > 0$ (see Exercise 1.41). As the first method is easier to apply, we shall adopt it to calculate the accumulated value over a non-integral compounding period, unless otherwise stated.

At the same nominal rate of interest, the more frequent the interest is paid, the faster the accumulated amount grows. For example, assuming the nominal rate of interest to be 5% and the principal to be \$1,000, the accumulated amounts after 1 year under several different compounding frequencies are given in Table 1.2.

Note that when the compounding frequency m increases, the accumulated amount tends to a limit. Let $\bar{r}$ denote the nominal rate of interest for which compounding is made over infinitely small intervals (i.e., $m \rightarrow \infty$ so that $\bar{r} = r^{(\infty)}$). We call this compounding scheme **continuous compounding**. For practical purposes, daily compounding is very close to continuous compounding.

From the well-known limit theorem (see Appendix A.1) that

$$\lim_{m \rightarrow \infty} \left[1 + \frac{\bar{r}}{m} \right]^m = e^{\bar{r}} \quad (1.8)$$

for any constant $\bar{r}$, we conclude that, for continuous compounding, the accumulation function (see (1.6)) is

$$a(t) = \lim_{m \rightarrow \infty} \left[1 + \frac{\bar{r}}{m} \right]^{mt} = \left[\lim_{m \rightarrow \infty} \left(1 + \frac{\bar{r}}{m} \right)^m \right]^t = e^{\bar{r}t}. \quad (1.9)$$

We call $\bar{r}$ the **continuously compounded rate of interest**. Equation (1.9) provides the accumulation function of the continuously compounding scheme at nominal rate of interest $\bar{r}$.

Table 1.2: Accumulated amount for a principal of \$1,000 with nominal interest rate of 5% per annum

Frequency of interest payment	m	Accumulated amount (\$)
Yearly	1	1,050.00
Quarterly	4	1,050.95
Monthly	12	1,051.16
Daily	365	1,051.27

1.4 Effective Rate of Interest

As Table 1.2 shows, the accumulated amount depends on the compounding frequency. Hence, comparing two investment schemes by just referring to their nominal rates of interest without taking into account their compounding frequencies may be misleading. Different investment schemes must be compared on a common basis. To this end, the measure called the **effective rate of interest** is often used. The annual effective rate of interest for year t , denoted by $i(t)$, is the ratio of the amount of interest earned in a year, from time $t - 1$ to time t , to the accumulated amount at the beginning of the year (i.e., at time $t - 1$). It can be calculated by the following formula

$$i(t) = \frac{I(t)}{A(t-1)} = \frac{A(t) - A(t-1)}{A(t-1)} = \frac{a(t) - a(t-1)}{a(t-1)}. \quad (1.10)$$

For the simple-interest method, we have

$$i(t) = \frac{(1 + rt) - (1 + r(t-1))}{1 + r(t-1)} = \frac{r}{1 + r(t-1)},$$

which decreases when t increases.

For the compound-interest method with annual compounding (i.e., $m = 1$), we have (denoting $r^{(1)} = r$)

$$i(t) = \frac{(1 + r)^t - (1 + r)^{t-1}}{(1 + r)^{t-1}} = r,$$

which is the nominal rate of interest and does not vary with t . When m -compounding is used, the effective rate of interest is

$$i(t) = \frac{\left[1 + \frac{r^{(m)}}{m}\right]^{tm} - \left[1 + \frac{r^{(m)}}{m}\right]^{(t-1)m}}{\left[1 + \frac{r^{(m)}}{m}\right]^{(t-1)m}} = \left[1 + \frac{r^{(m)}}{m}\right]^m - 1, \quad (1.11)$$

which again does not vary with t . Note that when $m > 1$,

$$\left[1 + \frac{r^{(m)}}{m}\right]^m - 1 > r^{(m)},$$

so that the effective rate of interest is larger than the nominal rate of interest.

For continuous compounding, we have

$$i(t) = \frac{\exp(\bar{r}t) - \exp[\bar{r}(t-1)]}{\exp[\bar{r}(t-1)]} = e^{\bar{r}} - 1, \quad (1.12)$$

which again does not vary with t .

As the effective rate of interest for the compound-interest method does not vary with t , we shall simplify the notation and denote $i \equiv i(t)$, which is given by (1.11) or (1.12). The effective rate of interest shows how much interest is earned in one year with a 1-unit investment. It is an appropriate *standardized* measure to compare interest-accumulation schemes with different compounding frequencies and nominal rates of interest.

Example 1.7: Consider two investment schemes A and B. Scheme A offers 12% interest with annual compounding. Scheme B offers 11.5% interest with monthly compounding. Calculate the effective rates of interest of the two investments. Which scheme would you choose?

Solution: The effective rate of interest of Scheme A is equal to its nominal rate of interest, i.e., 12%. The effective rate of interest of Scheme B is

$$\left[1 + \frac{0.115}{12}\right]^{12} - 1 = 12.13\%.$$

Although Scheme A has a higher nominal rate of interest, Scheme B offers a higher effective rate of interest. Hence, while an investment of \$100 in Scheme A will generate an interest of \$12 after one year, a similar investment in Scheme B will generate an interest of \$12.13 over the same period. Thus, Scheme B is preferred. ■

Another advantage of the effective rate of interest is that, for investments that extend beyond one year, the calculation of the accumulated amount can be based on the effective rate without reference to the nominal rate.

Example 1.8: For the investment schemes in Example 1.7, calculate the accumulated amount after 10 years on a principal of \$1,000.

Solution: The accumulated amount after 10 years for Scheme A is

$$1,000 (1 + 0.12)^{10} = \$3,105.85,$$

and that for Scheme B is

$$1,000 (1 + 0.1213)^{10} = \$3,142.09.$$

Note that in the above example, the accumulated amount of Scheme B is calculated without making use of its nominal rate. ■

Example 1.9: Solve the problem in Example 1.8 for an investment horizon of 12 years and 3 months.

Solution: The time frame of the investment is 12.25 years. Thus, the accumulated amount for Scheme A is

$$1,000 (1.12)^{12.25} = \$4,007.94,$$

and that for Scheme B is

$$1,000 (1.1213)^{12.25} = \$4,065.30.$$

Note that the calculation of the accumulated amount of Scheme B is done without using the fact that its compounding frequency is monthly. Indeed, the use of the effective rate of interest enables us to treat all investments *as if* the interest is credited only once a year and the compounding formula (1.4) is used for any $t \geq 0$, with the nominal rate r replaced by the effective rate i .



While $i(t)$ defined in (1.10) is a 1-period effective rate, the concept can be generalized to a n -period effective rate. Let us denote $i(t, t + n)$ as the annual effective rate of interest in the period t to $t + n$, for an integer $n > 1$. Thus, the amount $a(t)$ at time t compounded annually at the rate of $i(t, t + n)$ per year accumulates to $a(t + n)$ at time $t + n$, where

$$a(t + n) = a(t)[1 + i(t, t + n)]^n, \quad (1.13)$$

from which we obtain¹

$$i(t, t + n) = \left[\frac{a(t + n)}{a(t)} \right]^{\frac{1}{n}} - 1. \quad (1.14)$$

We now consider investment intervals shorter than one year. Suppose $\Delta t < 1$, the effective rate of interest in the period t to $t + \Delta t$, denoted by $i(t, t + \Delta t)$ can be defined as

$$i(t, t + \Delta t) = \frac{a(t + \Delta t) - a(t)}{a(t)}. \quad (1.15)$$

Note that this is an effective rate over a period of $\Delta t < 1$ and is *not annualized*.

Example 1.10: Let the accumulation function be $a(t) = 0.01t^2 + 0.1t + 1$. Compute the annual effective rate of interest for the first 2 years and for the next 3 years. What is the effective rate of interest for the second half of year 3?

¹We may use this formula to compute the annualized effective rate of interest over any investment horizon of $n > 1$, which is not necessarily an integer.

Solution: We have

$$a(2) = 0.01(2)^2 + 0.1(2) + 1 = 1.24,$$

and similarly $a(5) = 1.75$. Thus, from (1.14) the annual effective rate of interest over the first 2 years is

$$i(0, 2) = \left[\frac{a(2)}{a(0)} \right]^{\frac{1}{2}} - 1 = (1.24)^{0.5} - 1 = 11.36\%,$$

and the annual effective rate of interest over the next 3 years is

$$i(2, 5) = \left[\frac{a(5)}{a(2)} \right]^{\frac{1}{3}} - 1 = \left[\frac{1.75}{1.24} \right]^{\frac{1}{3}} - 1 = 12.17\%.$$

To compute the effective rate of interest for the second half of year 3, we note that $a(2.5) = 1.3125$ and $a(3) = 1.39$, so that the answer is

$$i(2.5, 3) = \frac{a(3) - a(2.5)}{a(2.5)} = \frac{1.39 - 1.3125}{1.3125} = 5.90\%.$$

Note that this is an effective rate of interest over a half-year period. In contrast, $i(0, 2)$ and $i(2, 5)$ are annualized effective rates over a period of 2 years and 3 years, respectively.



1.5 Rates of Discount

So far we have assumed that the interest of a loan or investment is paid at the end of the period. There are, however, many financial transactions for which the interest is paid or deducted up-front. Indeed, a popular way of raising a short-term loan is to sell a financial security at a price less than the face value. Upon the maturity of the loan, the face value is repaid. For example, a Treasury Bill is a discount instrument.

For a discount security, the shortfall between the sale price and the face value is called the **discount** and it represents the interest of the loan. In such cases the **nominal principal (face value)** has to be adjusted to take account of the interest deducted. This adjustment will affect the rate of interest, as opposed to the **rate of discount** that is quoted. Thus, if the lender of a loan requires the interest, which is calculated at the quoted annual rate of discount, to be deducted from the principal at the time the proceed is released, and if the loan period is one year, the **effective principal** of the loan after the interest is deducted is

$$\text{Effective principal} = \text{Nominal principal} \times (1 - \text{Rate of discount}).$$

Note that the effective principal is $A(0)$ and the nominal principal is $A(1)$. If the quoted rate of discount is d , we have

$$A(0) = A(1)(1 - d), \quad (1.16)$$

and

$$I(1) = A(1) - A(0) = A(1)d.$$

While the rate of discount is often quoted for such securities, it is important to compare it against the equivalent effective rate of interest i over the period of the discount instrument, which is given by

$$i = \frac{A(1) - A(0)}{A(0)} = \frac{A(1)d}{A(1)(1 - d)} = \frac{d}{1 - d}, \quad (1.17)$$

from which we have

$$d = \frac{i}{1 + i}.$$

Combining (1.4) and (1.17), we can see that

$$\begin{aligned} a(t) &= (1 + i)^t \\ &= \left[1 + \frac{d}{1 - d}\right]^t \\ &= (1 - d)^{-t}, \end{aligned}$$

which is the accumulated value of 1 at time t at the rate of discount d .

It should be noted that the effective rate of interest i is always higher than the rate of discount d . For example, if the rate of discount is 6% and the redemption of the security is to be made after 1 year, the effective rate of interest is

$$\frac{0.06}{1 - 0.06} = 6.38\%.$$

When the loan period is less than 1 year, we should first calculate the rate of interest over the period of the loan and then calculate the effective rate of interest using the principle of compounding. Thus, suppose the period of loan is $\frac{1}{m}$ of a year, we denote the **nominal rate of discount**, which is the discount quoted in annual term for the $\frac{1}{m}$ -year instrument, by $d^{(m)}$. As the face value is to be repaid $\frac{1}{m}$ of a year later, the nominal principal is $A\left(\frac{1}{m}\right)$ and the interest deducted is

$$I\left(\frac{1}{m}\right) = A\left(\frac{1}{m}\right) \times d^{(m)} \times \frac{1}{m},$$

so that the effective principal is

$$A(0) = A\left(\frac{1}{m}\right) - I\left(\frac{1}{m}\right) = A\left(\frac{1}{m}\right) \left[1 - \frac{d^{(m)}}{m}\right],$$

from which we obtain the rate of interest charged over the $\frac{1}{m}$ -year period as

$$\frac{A\left(\frac{1}{m}\right) - A(0)}{A(0)} = \frac{\frac{d^{(m)}}{m}}{1 - \frac{d^{(m)}}{m}} = \frac{d^{(m)}}{m - d^{(m)}}.$$

Hence, the **annualized equivalent nominal rate of interest** is

$$r^{(m)} = m \times \frac{d^{(m)}}{m - d^{(m)}} = \frac{d^{(m)}}{1 - \frac{d^{(m)}}{m}}, \quad (1.18)$$

and the annual effective rate of interest is

$$i = \left[1 + \frac{r^{(m)}}{m}\right]^m - 1 = \left[1 + \frac{d^{(m)}}{m - d^{(m)}}\right]^m - 1 = \left[1 - \frac{d^{(m)}}{m}\right]^{-m} - 1. \quad (1.19)$$

As Treasury Bills that mature in less than a year are discount instruments, their effective rates should be computed in the above manner.

To compute the accumulation function $a(t)$ for a discount instrument with maturity of $\frac{1}{m}$ year ($m > 1$), we observe, from (1.19), that

$$\begin{aligned} a(t) &= (1 + i)^t \\ &= \left[1 - \frac{d^{(m)}}{m}\right]^{-mt}, \quad \text{for } t = 0, \frac{1}{m}, \frac{2}{m}, \dots \end{aligned} \quad (1.20)$$

Furthermore, from (1.19) we have

$$1 - \frac{d^{(m)}}{m} = (1 + i)^{-\frac{1}{m}}, \quad (1.21)$$

so that from (1.18) we conclude

$$r^{(m)} = (1 + i)^{\frac{1}{m}} d^{(m)}. \quad (1.22)$$

For a discount security with a loan period longer than one year, (1.16) can be modified to

$$A(0) = A(t)(1 - dt), \quad 1 < t,$$

provided $dt < 1$. Hence, we have

$$A(t) = \frac{A(0)}{1 - dt}, \quad dt < 1 < t,$$

and

$$a(t) = \frac{1}{1 - dt}, \quad dt < 1 < t, \quad (1.23)$$

which is the accumulation function for simple discount for $dt < 1 < t$.

Example 1.11: The discount rate of a 3-month Treasury Bill is 6% per annum. What is the annual effective rate of interest? What is the accumulated value of 1 in 2 years?

Solution: The rate of interest charged for the 3-month period is

$$\frac{0.06 \times \frac{1}{4}}{1 - 0.06 \times \frac{1}{4}} = 1.52\%.$$

Therefore, the equivalent nominal rate of interest compounded quarterly is

$$4 \times 0.0152 = 6.08\%,$$

and the annual effective rate of interest is

$$(1.0152)^4 - 1 = 6.22\%.$$

The accumulated value of 1 in 2 years is, using (1.20),

$$a(2) = \left[1 - \frac{0.06}{4}\right]^{-8} = 1.13.$$

Note that this can also be calculated as

$$(1 + i)^2 = (1.0622)^2 = 1.13.$$



Example 1.12: The discount rate of a 6-month Treasury Bill is 8% per annum, what is the annual effective rate of interest? What is the accumulated value of 1 in 3 years?

Solution: The rate of interest charged for the 6-month period is

$$\frac{0.08 \times \frac{1}{2}}{1 - 0.08 \times \frac{1}{2}} = 4.17\%.$$

Therefore, the annualized equivalent nominal rate of interest compounded semiannually is

$$2 \times 0.0417 = 8.34\%,$$

and the annual effective rate of interest is

$$(1.0417)^2 - 1 = 8.51\%.$$

The accumulated value of 1 in 3 years is

$$a(3) = \left[1 - \frac{0.08}{2}\right]^{-6} = 1.28,$$

which can also be computed as

$$(1 + i)^3 = (1.0851)^3 = 1.28.$$



1.6 Force of Interest

Given any accumulation function $a(t)$, the effective rate of interest over the time interval $(t, t + \Delta t)$ can be computed as $i(t, t + \Delta t)$ using (1.15), which measures how fast the investment accumulates in value at time t . The rate computed by (1.15), however, is not annualized. But if we divide $i(t, t + \Delta t)$ by Δt , we obtain the rate of interest of the investment *per unit time* in the interval $(t, t + \Delta t)$ (if t is in years, the rate of interest is per annum). A particularly interesting question is the *instantaneous* rate, which is obtained when Δt is infinitesimally small. Thus, we consider $\frac{i(t, t + \Delta t)}{\Delta t}$ when Δt tends to zero, which is given by

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{i(t, t + \Delta t)}{\Delta t} &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\frac{a(t + \Delta t) - a(t)}{a(t)} \right] \\ &= \frac{1}{a(t)} \lim_{\Delta t \rightarrow 0} \left[\frac{a(t + \Delta t) - a(t)}{\Delta t} \right] \\ &= \frac{a'(t)}{a(t)}, \end{aligned} \tag{1.24}$$

where $a'(t)$ is the derivative of $a(t)$ with respect to t . Thus, we define

$$\delta(t) = \frac{a'(t)}{a(t)}, \quad (1.25)$$

which is called the **force of interest**. As (1.24) shows, the force of interest is the instantaneous rate of increase of the accumulated amount, $a'(t)$, as a percentage of the accumulated amount at time t , $a(t)$.

Given $a(t)$, the force of interest $\delta(t)$ can be computed using (1.25). Now we show that the computation can be reversed, i.e., given $\delta(t)$ we can compute $a(t)$. First, we note that (1.25) can be written as

$$\delta(t) = \frac{d \ln a(t)}{dt},$$

from which we have

$$\begin{aligned} \int_0^t \delta(s) ds &= \int_0^t d \ln a(s) \\ &= \ln a(s) \Big|_0^t \\ &= \ln a(t) - \ln a(0) \\ &= \ln a(t), \end{aligned}$$

as $a(0) = 1$. Hence, we conclude

$$a(t) = \exp \left(\int_0^t \delta(s) ds \right). \quad (1.26)$$

Equation (1.26) provides the method to calculate the accumulation function given the force of interest. In the case when the force of interest is constant (not varying with t), we denote $\delta(t) \equiv \delta$, and the integral in (1.26) becomes δt . Thus, we have

$$a(t) = e^{\delta t}. \quad (1.27)$$

Comparing (1.27) with (1.9) we can see that if the force of interest is constant, it is equal to the continuously compounded rate of interest, i.e., $\bar{r} = \delta$.

We now derive the force of interest for the simple- and compound-interest methods. For the simple-interest method, we obtain, from (1.2),

$$a'(t) = r,$$

so that

$$\delta(t) = \frac{r}{1 + rt}, \quad \text{for } t \geq 0. \quad (1.28)$$

Hence, $\delta(t)$ decreases as t increases. In other words, the instantaneous rate of interest as a percentage of the accumulated amount drops with time.

For the compound-interest method, we have, from (1.4) and Appendix A.6 (with i replacing r irrespective of the compounding frequency),

$$a'(t) = (1 + i)^t \ln(1 + i),$$

so that

$$\delta(t) = \frac{(1 + i)^t \ln(1 + i)}{(1 + i)^t} = \ln(1 + i). \quad (1.29)$$

Thus, $\delta(t)$ does not vary with time and we write $\delta(t) \equiv \delta$, from which equation (1.29) implies

$$e^\delta = 1 + i.$$

Comparing this with (1.12), we again conclude that the force of interest is the continuously compounded rate of interest.

Example 1.13: A fund accumulates at a simple-interest rate of 5%. Another fund accumulates at a compound-interest rate of 4%, payable yearly. When will the force of interest be the same for the two funds? After this time, which fund will have a higher force of interest?

Solution: From (1.28), the force of interest of the simple-interest fund at time t is

$$\delta(t) = \frac{0.05}{1 + 0.05t}.$$

From (1.29), the force of interest of the compound-interest fund is $\ln(1.04)$ at any time. The two funds have the same force of interest when

$$\frac{0.05}{1 + 0.05t} = \ln(1.04),$$

i.e.,

$$t = \frac{0.05 - \ln(1.04)}{0.05 \ln(1.04)} = 5.4967,$$

after which the force of interest of the simple-interest fund remains lower than that of the compound-interest fund.



Example 1.14: If a fund accumulates at force of interest $\delta(t) = 0.02t$, find the annual effective rate of interest over 2 years and 5 years.

Solution: From (1.26), we have

$$a(2) = \exp \left(\int_0^2 0.02s \, ds \right) = \exp \left(0.01s^2 \Big|_0^2 \right) = e^{0.04}.$$

We solve the annual effective rate of interest i over the 2-year period from the equation (compare this with (1.13))

$$(1 + i)^2 = e^{0.04}$$

to obtain

$$i = e^{0.02} - 1 = 2.02\%.$$

Similarly,

$$a(5) = \exp \left(0.01s^2 \Big|_0^5 \right) = e^{0.25},$$

so that the annual effective rate of interest i over the 5-year period satisfies

$$(1 + i)^5 = e^{0.25}$$

and we obtain

$$i = e^{0.05} - 1 = 5.13\%.$$



1.7 Present and Future Values

At the effective rate of interest i , a 1-unit investment today will accumulate to $(1 + i)$ units at the end of the year. In this respect, the accumulated amount $(1 + i)$ is also called the **future value** of 1 at the end of the year. Similarly, the future value of 1 at the end of year t is $(1 + i)^t$. This is the amount of money you can get t years later if you invest 1 unit today at the effective rate i .

Sometimes it may be desirable to find the initial investment that will accumulate to a targeted amount after a certain period of time. For example, a $\frac{1}{1+i}$ -unit payment invested today will accumulate to 1 unit at the end of the year. Thus, $\frac{1}{1+i}$ is called the **present value** of 1 to be paid at the end of year 1. Extending the time frame to t years, the present value of 1 due at the end of year t is $\frac{1}{(1+i)^t}$.

Example 1.15: Given $i = 6\%$, calculate the present value of 1 to be paid at (a) the end of year 1, (b) the end of year 5 and (c) 6.5 years.

Solution: (a) The present value of 1 to be paid at the end of year 1 is

$$\frac{1}{1 + 0.06} = 0.9434.$$

The answers to (b) and (c) are, respectively,

$$\frac{1}{(1 + 0.06)^5} = 0.7473$$

and

$$\frac{1}{(1 + 0.06)^{6.5}} = 0.6847.$$



Example 1.16: An insurance agent offers a policy that pays a lump sum of \$50,000 five years later. If the effective rate of interest is 8%, how much would you pay for the plan?

Solution: The *fair* amount to pay for this policy is the present value of the lump sum, which is equal to

$$\frac{50,000}{(1.08)^5} = \$34,029.16.$$



Example 1.17: A person wants to accumulate \$100,000 eight years from today to sponsor his son's education. If an investment plan offers him 8% compounded monthly, what amount must he invest today?

Solution: We first calculate the effective rate of interest, which is

$$\left[1 + \frac{0.08}{12}\right]^{12} - 1 = 8.30\%.$$

The amount required today is

$$\frac{100,000}{(1.083)^8} = \$52,841.16.$$



We now denote

$$v = \frac{1}{1 + i}, \tag{1.30}$$

which is the **present value** of 1 due 1 year later. It is also called the **discount factor**, as multiplying a payment at the end of the year by v gives its present value. Combining (1.17) and (1.30), we obtain

$$d = iv, \tag{1.31}$$

so that the present value of i is d . Also,

$$v + d = \frac{1}{1+i} + \frac{i}{1+i} = 1, \quad (1.32)$$

which says that a unit payment at time 1 is the sum of its present value and discount.

Since $a(1) = 1 + i$, (1.30) can also be written as

$$v = \frac{1}{a(1)},$$

which says that the present value of 1 due 1 year later is the reciprocal of the accumulation function evaluated at time $t = 1$. For a general time t , we denote $v(t)$ as the present value of 1 to be paid at time t . Then

$$v(t) = \frac{1}{(1+i)^t} = \frac{1}{a(t)}, \quad (1.33)$$

which is the discount factor for payments at time t .

The above calculations apply to the case of the compound-interest method with a constant effective rate of interest. For a general accumulation function $a(\cdot)$, the discount factor for payments at time t is

$$v(t) = \frac{1}{a(t)}.$$

Thus, when the simple-interest method is used, the present value of 1 due t years later at the nominal rate of interest r is

$$v(t) = \frac{1}{a(t)} = \frac{1}{1+rt}. \quad (1.34)$$

Example 1.18: Find the sum of the present values of two payments of \$100 each to be paid at the end of year 4 and 9, if (a) interest is compounded semiannually at the nominal rate of 8% per year, and (b) the simple-interest method at 8% per year is used.

Solution: We first calculate the discount factors $v(4)$ and $v(9)$. For case (a), the effective rate of interest is

$$(1.04)^2 - 1 = 0.0816,$$

so that

$$v(4) = \frac{1}{(1.0816)^4} = 0.7307$$

and

$$v(9) = \frac{1}{(1.0816)^9} = 0.4936.$$

Hence, the present value of the two payments is

$$100(0.7307 + 0.4936) = \$122.43.$$

For case (b), we have

$$v(4) = \frac{1}{1 + 0.08 \times 4} = 0.7576$$

and

$$v(9) = \frac{1}{1 + 0.08 \times 9} = 0.5814,$$

so that the present value of the two payments is

$$100(0.7576 + 0.5814) = \$133.90.$$

Note that as the accumulation function for simple interest grows slower than that for compound interest, the present value of the simple-interest method is higher.



We now consider a payment of 1 at a future time τ . What is the future value of this payment at time $t > \tau$? The answer to this question depends on how a payment at a future time accumulates with interest. Let us assume that any future payment starts to accumulate interest following the same accumulation function as a payment made at time 0.² As the 1-unit payment at time τ earns interest over a period of $t - \tau$ until time t , its accumulated value at time t is $a(t - \tau)$.

However, if we consider a different scenario in which the 1-unit amount at time τ has been accumulated from time 0 and is not a new investment, what is the future value of this amount at time t ? To answer this question, we first determine the invested amount at time 0, which is the present value of 1 due at time τ , i.e., $\frac{1}{a(\tau)}$. The future value of this investment at time t is then given by

$$\frac{1}{a(\tau)} \times a(t) = \frac{a(t)}{a(\tau)}.$$

²We adopt this assumption for the purpose of defining future values for future payments. It is not claimed that the current accumulation function applies to all future investments in practice.

We can see that the future values of the two investments, i.e., a 1-unit investment at time τ versus a 1-unit amount at time τ accumulated from time 0, are not necessarily the same. However, they are equal if

$$a(t - \tau) = \frac{a(t)}{a(\tau)} \quad (1.35)$$

for $t > \tau > 0$. Figure 1.1 illustrates the evaluation of the future values at time t of these two investments.

Note that compound-interest accumulation satisfies the condition (1.35). The principal at time τ accumulates interest at the rate of i per year, whether the principal is invested at time τ or is accumulated from the past. Specifically, we have

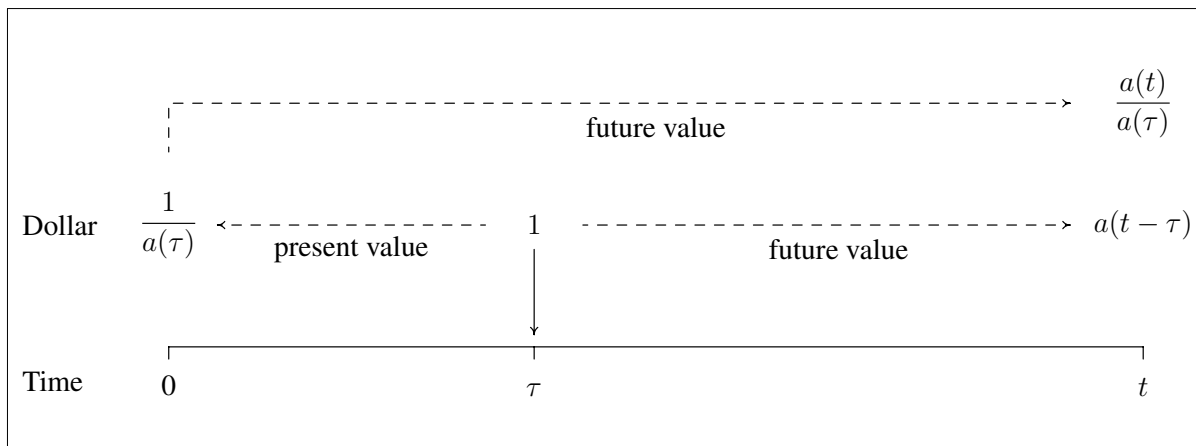
$$\frac{a(t)}{a(\tau)} = \frac{(1+i)^t}{(1+i)^\tau} = (1+i)^{t-\tau} = a(t - \tau).$$

In contrast, simple-interest accumulation does not satisfy condition (1.35). The future value at time t of a unit payment at time τ is $a(t - \tau) = 1 + r(t - \tau)$. However, we have

$$\frac{a(t)}{a(\tau)} = \frac{1 + rt}{1 + r\tau} = \frac{1 + r\tau + r(t - \tau)}{1 + r\tau} = 1 + \frac{r(t - \tau)}{1 + r\tau} < 1 + r(t - \tau) = a(t - \tau).$$

The discrepancy is due to the fact that only the principal of $\frac{1}{a(\tau)}$ invested at time 0 is paid interest in the period τ to t , although the principal has accumulated to 1 unit at time τ . On the other hand, if 1 unit of payment is invested at time τ , this is the amount that generates interest in the period τ to t .

Figure 1.1: Present and future values of unit payment at time τ



Example 1.19: Let $a(t) = 0.02t^2 + 1$. Calculate the future value of an investment at $t = 5$ consisting of a payment of 1 now and a payment of 2 at $t = 3$. You may assume that future payments earn the current accumulation function.

Solution: The future value at time 5 is

$$a(5) + 2 \times a(3) = [0.02(5)^2 + 1] + 2[0.02(3)^2 + 1] = 3.66.$$

■

Example 1.20: Let $\delta(t) = 0.01t$. Calculate the future value of an investment at $t = 5$ consisting of a payment of 1 now and a payment of 2 at $t = 2$. You may assume that future payments earn the current accumulation function.

Solution: We first derive the accumulation function from the force of interest, which is

$$a(t) = \exp\left(\int_0^t 0.01s \, ds\right) = \exp(0.005t^2).$$

Thus, $a(5) = \exp(0.125) = 1.1331$ and $a(3) = \exp(0.045) = 1.0460$, from which we obtain the future value of the investment as

$$a(5) + 2 \times a(3) = 1.1331 + 2(1.0460) = 3.2251.$$

■

1.8 Equation of Value

Consider a stream of cash flows occurring at different times. The present value of the cash flows is equal to the sum of the present values of each payment. To fix ideas, assume the payments are of values C_j occurring at time $j = 0, 1, \dots, n$, and compound-interest accumulation is adopted. If the annual effective rate of interest is i with corresponding discount factor v , the present value P of the cash flows is given by

$$P = \sum_{j=0}^n C_j v^j \quad (1.36)$$

and the future value F of the cash flows at time n is

$$F = (1 + i)^n P = (1 + i)^n \left(\sum_{j=0}^n C_j v^j \right) = \sum_{j=0}^n C_j (1 + i)^{n-j}. \quad (1.37)$$

Thus, equations (1.36) and (1.37) involve the quantities P , F , C_j , i and n .³ We call these the **equations of value**. Given C_j , i and n , P and F can be calculated straightforwardly. Alternatively, if C_j and i are given, we can calculate the time n given the present value P or future value F . The examples below illustrate this point.

Example 1.21: At the annual effective rate of interest i , when will an initial principal be doubled?

Solution: This requires us to solve for n from the equation (note that $C_j = 0$ for $j > 0$, and we let $C_0 = 1$ and $F = 2$)

$$(1 + i)^n = 2,$$

from which

$$n = \frac{\ln(2)}{\ln(1 + i)}.$$

Thus, n is generally not an integer but can be solved exactly from the above equation.

To obtain an approximate solution for n , we note that $\ln(2) = 0.6931$ so that

$$n = \frac{0.6931}{i} \times \frac{i}{\ln(1 + i)}.$$

We approximate the last fraction in the above equation by taking $i = 0.08$ to obtain

$$n \approx \frac{0.6931}{i} \times 1.0395 = \frac{0.72}{i}.$$

Thus, n can be calculated approximately by dividing 0.72 by the effective rate of interest. This is called the **rule of 72**. It provides a surprisingly accurate approximation to n for a wide range of values of i . For example, when $i = 2\%$, the approximation gives $n = 36$ while the exact value is 35. When $i = 14\%$, the approximate value is 5.14 while the exact value is 5.29.



Example 1.22: How long will it take for \$100 to accumulate to \$300 if interest is compounded quarterly at the nominal rate of 6% per year?

³Note that as we are assuming compound interest, (1.35) holds. Thus, we can compute the future value of future payments as the future value of their corresponding present values, as in equation (1.37).

Solution: Over one quarter, the interest rate is 1.5%. With $C_0 = \$100$, $C_j = 0$ for $j > 0$, and $F = \$300$, from (1.37) the equation of value is (n is in quarters)

$$300 = 100(1.015)^n,$$

so that

$$n = \frac{\ln(3)}{\ln(1.015)} = 73.79 \text{ quarters,}$$

i.e., 18.45 years.



Another problem is the solution of the rate of interest that will give rise to a targeted present value or future value with given cash flows. The example below illustrates this point.

Example 1.23: A student takes out a tuition loan of \$15,000, and is required to pay back with a step-up payment \$7,000 in year 1 and \$8,500 in year 2. What is the effective rate of interest she is charged?

Solution: The equation of value is, from (1.36),

$$15,000 = 7,000v + 8,500v^2.$$

Solving for the quadratic equation (see Appendix A.3), we obtain, after dropping the negative root,

$$v = \frac{-7 + \sqrt{7^2 + 4 \times 8.5 \times 15}}{2 \times 8.5} = 0.979,$$

which implies

$$i = \frac{1}{0.979} - 1 = 2.14\%.$$



Note that the solution of the above problem can be obtained analytically by solving a quadratic equation, as there are only two payments. When there are more payments, the solution will require the use of numerical methods, the details of which can be found in Chapter 4.

Example 1.24: A savings fund requires the investor to pay an equal amount of installment each year for 3 years, with the first installment to be paid immediately. At the end of the 3 years, a lump sum will be paid back to the investor. If the effective interest rate is 5%, what is the amount of the installment such that the investor can receive a lump sum of \$10,000?

Solution: Let k be the installment. From (1.37), the equation of value is

$$10,000 = k[(1.05)^3 + (1.05)^2 + 1.05] = 3.31k,$$

so that the installment is

$$k = \frac{10,000}{3.31} = \$3,021.03.$$



It can be observed that if the horizon of the savings fund is long, the computation of the above can be quite tedious. In the next chapter, we will see how the computation can be made easier for installments over a long horizon.

1.9 Summary

1. The amount function and accumulation function trace the accumulation of an investment over a period of time. The interest incurred in each period can be calculated from the amount function.
2. The simple-interest and compound-interest methods are two commonly used methods for calculating interest incurred. For the compound-interest method, the frequency of compounding is an important determinant of how interest accrues. To compare schemes with different compounding frequencies, the effective rate of interest should be used.
3. Discount instruments are popular in raising short-term loans. The effective rate of interest is always higher than the discount rate as the effective principal is less than the nominal principal or face value.
4. The force of interest is the instantaneous rate of interest as a percentage of the accumulated amount. It is a useful measure of how investment accumulates, especially when the rate of interest is time varying. If the force of interest is constant, it is equal to the continuously compounded rate of interest.
5. The present value of a stream of payments is equal to the sum of the present values of each payment. It aggregates the payments by properly weighting each of them by a discount factor that takes account of the differences in the time the payments are made.

6. The equation of value links the present value or the future value with the installments, the period of the payments and the interest rate. It can be used to solve for one variable given the others.
7. Table 1.3 summarizes the accumulated value and present value for various accumulation methods.

Exercises

- 1.1 An investor invests \$20,000 into a fund for 4 years. The annual nominal interest rate remains at 8% in each year although it is convertible semiannually in the first year, quarterly in the second year, bi-monthly in the third year, and monthly in the fourth year. Find the accumulated value of the fund at the end of the fourth year.
- 1.2 For the compound-interest method over fraction of a compounding period, show that the second method described in Section 1.3 provides a larger accumulation amount over the first method.
- 1.3 If $A(4) = 1,200$ and $i(t) = 0.01t^2$, find $I(5)$ and $A(6)$.
- 1.4 It is given that $A(0) = 300$, $I(1) = 5$, $I(2) = 7$, $I(3) = 9$ and $I(4) = 14$. Find $A(3)$ and $i(4)$.
- 1.5 Find the accumulated value of \$1,000 at the end of the fourth year
 - (a) if the simple discount rate is 6% per annum,
 - (b) if the simple interest rate is 6% per annum,
 - (c) if the effective rate of interest is 6% per annum,
 - (d) if the annual nominal interest rate is 6% payable quarterly,
 - (e) if the annual nominal discount rate is 6% compounded monthly,
 - (f) if the constant force of interest is 6% per annum.
- 1.6 Assume that $a(t) = \ln(0.5t^2 + e) + 0.05t^{0.3}$, find
 - (a) $i(2)$ and $i(3)$,
 - (b) the amount of interest earned in the third period if the principal is \$1,200.

Table 1.3: Summary of accumulated value and present value formulas (all rates quoted per year)

Accumulation method	Freq of conversion per year	Rate of interest or discount	Future value of 1 at time t (in years): $a(t)$	Present value of 1 due at time t (in years): $1/a(t)$	Equations in book	Remarks
Compound interest	1	$r (= i)$	$(1 + r)^t$	$(1 + r)^{-t}$	(1.4)	also applies to non-integer $t > 0$
Compound interest	m	$r^{(m)}$	$\left[1 + \frac{r^{(m)}}{m}\right]^{mt}$	$\left[1 + \frac{r^{(m)}}{m}\right]^{-mt}$	(1.6)	$m > 1$ if compounding more frequent than annually
Compound interest	∞	$\bar{r}$	$e^{\bar{r}t}$	$e^{-\bar{r}t}$	(1.9)	$\bar{r}$ is the constant force of interest
Compound discount	1	d	$(1 - d)^{-t}$	$(1 - d)^t$	(1.4), (1.7)	
Compound discount	m	$d^{(m)}$	$\left[1 - \frac{d^{(m)}}{m}\right]^{-mt}$	$\left[1 - \frac{d^{(m)}}{m}\right]^{mt}$	(1.19)	$m > 1$ for loan shorter than 1 year
Simple interest		r	$1 + rt$	$(1 + rt)^{-1}$	(1.2)	
Simple discount		d	$(1 - dt)^{-1}$	$1 - dt$	(1.23)	

1.7 It is given that $i(t) = 0.01 + 0.005t$.

- (a) Find $a(t)$ for t being an integer.
- (b) If the principal is \$100, find the total amount of interest earned in year 3, 4 and 5.

1.8 Which of the following is a valid accumulation function?

- (a) $a(t) = t^2 + 2t + 2$,
- (b) $a(t) = t^2 - 2t + 1$,
- (c) $a(t) = t^2 + 2t + 1$.

1.9 If $d^{(4)} = 0.0985$, find

- (a) i ,
- (b) $d^{(2)}$,
- (c) $r^{(4)}$,
- (d) δ ,
- (e) $a(t)$.

1.10 A security is sold at a discount of 7%, compounded semiannually. Find the effective rate of interest.

1.11 Prove that $i - d = id$ by using formulas (1.30) and (1.31). What does this relation tell you?

1.12 \$1,500 is deposited into a savings account that pays 3.5% interest with monthly compounding. What is the accumulated amount after four and a quarter years? What is the amount of interest earned over this period?

1.13 \$10,000 is deposited into a savings account that earns 3% per annum compounded quarterly.

- (a) How much interest will be credited in the second month?
- (b) How much interest will be credited in the second year?

1.14 Using the rule of 72, calculate the approximate number of years for an initial principal to double if the annual effective rate of interest is 4%. Compare your answer with the true value.

1.15 Find the present value of \$500 to be paid at the end of 28 months.

- (a) if the nominal interest rate is 6% convertible monthly,
- (b) if the nominal discount rate is 6% compounded every 4 months,
- (c) if the nominal interest rate is 4% compounded semiannually,
- (d) if the nominal discount rate is 5% payable annually.

Apply the simple-interest method for the remaining fraction of the conversion period if necessary.

1.16 When does the rule of 72 hold exactly?

1.17 You wish to borrow \$10,000. The following is the information you gathered from two lenders:

- (a) Lender A charges 7% compounded quarterly,
- (b) Lender B charges an annual effective rate of interest of 7.25%.

Which lender would you choose?

1.18 A Treasury Bill for \$100 is purchased for \$95 four months before due. Find the nominal rate of discount convertible three times a year earned by the purchaser.

1.19 You are given the following stream of cash flows:

t	C_t
0	950
1	800
2	150
3	400
4	120

Find the present value and the future value (after 5 years) of the cash flows evaluated at $i = 2\%$.

1.20 (a) Extend formula (1.36) to the case when the interest rate is not constant over time.

(b) It is given that

t	$v(t)$
1	0.98
2	0.95
3	0.92
4	0.90

Find the present value of the following stream of payments:

t	C_t (\$)
0	1,000
1	800
2	600
3	400
4	200

1.21 Let $v(t) = \frac{20}{20+t}$.

- (a) Find $i(1)$, $i(2)$ and $i(3)$.
 (b) How is the interest credited?

1.22 Beda wishes to accumulate \$3,000 in a fund at the end of 4 years. He makes deposits of \$ K , \$2 K , \$3 K at the beginning of year 1, 2 and 3, respectively. Find the value of K if the fund earns

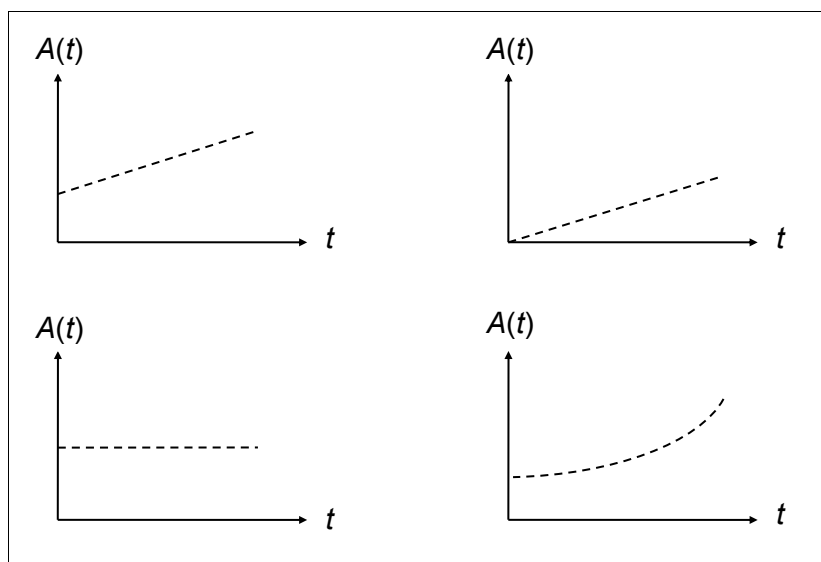
- (a) simple interest of 5% per year,
 (b) compound interest of 5% per year.

1.23 Eddy deposits \$2,000 into a savings account. The bank credits interest at a rate of i convertible yearly for the first 4 years and a nominal rate of $4i$ convertible quarterly thereafter. If the accumulated value of the account is \$2,918.70 after 6 years, find the accumulated value of the account after 8.5 years.

1.24 Ada pays \$1,000 today and \$1,200 after 2 years to Andrew in exchange for a payment of \$2,198 one year from today.

- (a) Set up an equation of value with i as unknown by equating the future values of the two cash-flow streams.
 (b) Solve the equation of value obtained in (a).
 (c) What can you observe?

- 1.25 \$2,000 is put into an account which credits interest at 4% per annum and \$1,500 is put into another account which credits interest at 6% per annum. How long does it take for the balances of the two accounts to be equal? Apply the compound-interest method for the remaining fraction of the conversion period.
- 1.26 \$2,000 is put into an account which credits interest at 4% per annum and \$1,500 is put into another account which credits interest at 6% per annum. How long does it take for the balances of the two accounts to be equal? Apply the simple-interest method for the remaining fraction of the conversion period. Compare your answer with that in Exercise 1.25.
- 1.27 Which of the following are valid graphs for the amount function $A(t)$?



- 1.28 Irene invests \$5,000 in an account that credits interest during the first two years at a nominal interest rate of $r^{(2)}$ convertible semiannually. During the third year, the account earns interest at a nominal discount rate of $d^{(4)}$ convertible quarterly. At the end of the third year, the fund has accumulated to \$5,375.70. If $r^{(2)} = d^{(4)} = x$, find x .
- 1.29 Betty deposits \$200 at the beginning of each year for 10 years into an account which credits simple interest at an annual interest rate of i payable every year. Find i if the accumulated value is \$2,440 after 10 years.
- 1.30 You are given the following information:

t	$v(t)$
1	0.97
2	0.94
3	0.91
4	0.88
5	0.85

- (a) You want to accumulate \$10,000 five years from today. What amount should you invest today?
- (b) You want to accumulate \$10,000 five years from today by two installments of equal amount, to be paid at the beginning of the first and the third year. How much are the two installments? You may follow the assumption in Section 1.7 for the computation of the future value of future payments.
- 1.31 The present value of two payments, the first payment of \$500 paid at the end of n years, and the second payment of \$1,000 at the end of $2n$ years, is \$1,158. If $i = 4\%$, find the value of n .
- 1.32 Albert takes out a loan of \$30,000, and is required to pay back with a payment of \$20,000 at the end of the second year, and \$11,000 at the end the fourth year. Interest is payable quarterly. What is the nominal rate of interest charged on the loan?
- 1.33 Let $a(t) = \frac{1}{1 - 0.01t}$ for $0 \leq t < 100$.
- (a) Draw the graph of $v(t)$ for $0 \leq t < 100$.
- (b) How is the interest credited?
- 1.34 Let $A(t) = t^2 + 2t + 4$. Find $\delta(5)$.
- 1.35 Which of the following are properties of the force of interest $\delta(t)$?
- (a) It is non-negative.
- (b) It is increasing with t .
- (c) It is continuous.
- 1.36 Assume that $\delta(t) = \frac{1}{10(1+t)^3}$ and $A(0) = 100$. Find the amount of interest earned in the fifth year.

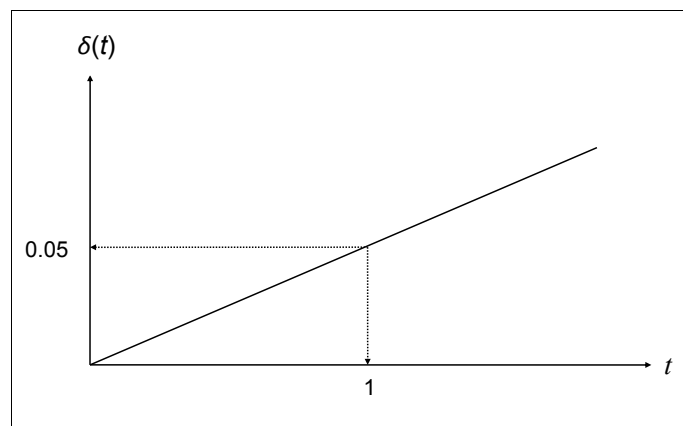
1.37 Calculate the future value of an investment at $t = 5$ consisting of a payment of 1 at $t = 2$ and a payment of 3 at $t = 4$, if

(a) $a(t) = 1 + 0.05t$,

(b) $a(t) = (1.05)^t$,

(c) $\delta(t) = 0.05t$.

1.38 In Fund P, money accumulates at a force of interest $\delta(t)$ whose graph is given below.



(a) Find $a(5)$.

(b) Fund Q accumulates at a constant force of interest δ_Q . If the value of the two funds are equal at $t = 0$ and $t = 5$, find δ_Q .

1.39 You are given the following stream of cash flows:

t	0.5	1	1.5	2	2.5	3
C_t	10	10	10	10	10	110

Find the present value of the cash flows given that the nominal rate of interest is 8% compounded half-yearly.

1.40 Certificates of deposit (CDs) are short- to medium-term investment instruments issued by banks which provides a fixed rate of interest for a period of time. CDs offer stability in interest earned but have penalty for early withdrawal.

A 1-year CD pays a nominal rate of interest of 8% compounded quarterly. You are offered two options of penalty for early withdrawal:

- (a) loss of 3-month interest;
- (b) a reduction in the nominal rate of interest to 6%.

If you wish to withdraw after 9 months, which option would you choose?

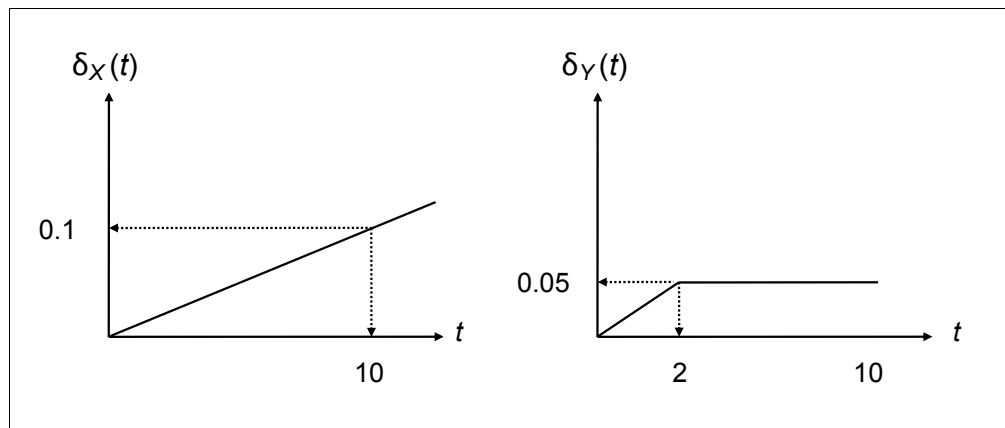
Advanced Problems

- 1.41 (a) For $c > 0$, let $f(x) = 1 + cx$ and $g(x) = (1 + c)^x$, where $0 \leq x \leq 1$.
- (1) Show that $f(0) = g(0)$, and $f(1) = g(1)$.
 - (2) By using the fact that $g(\cdot)$ is increasing in x , show that $f(x) \geq g(x)$ for $0 \leq x \leq 1$.
 - (3) Letting $c = r$ in (a)(2), what conclusion can you draw?
- (b) Let $[x]$ be the integer part of x . For examples, $[4.2] = 4$, $[9.99] = 9$ and $[1] = 1$. By using the result in (a)(2), show that for $t \geq 0$,

$$\left(1 + \frac{r^{(m)}}{m}\right)^{mt - [mt]} \leq 1 + r^{(m)} \left(\frac{mt - [mt]}{m}\right).$$

- (c) Prove that when the loan period is not an integer multiple of the compounding period, applying the simple-interest method for the remaining fraction of the conversion part always gives a larger accumulation function when compared with extending (1.6) to any $mt \geq 0$.

- 1.42 The following graphs plot the force of interest for Fund X and Fund Y for $0 \leq t \leq 10$.



Find the points in time at which the values of the accumulation functions of the two funds are equal.

1.43 Assume that the force of interest is constant over time. Let the constant force of interest be δ and the equivalent effective rate of interest and discount be i and d , respectively.

- (a) Express δ in terms of i .
- (b) Express δ in terms of d .
- (c) By using the Taylor's expansion

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

and the expressions in (a) and (b), show that

$$\delta = \frac{1}{2} \left[\frac{i+d}{1} - \frac{i^2-d^2}{2} + \frac{i^3+d^3}{3} - \frac{i^4-d^4}{4} + \cdots \right].$$

The formula above shows that δ is very close to the average of i and d .

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