

Discrete Mathematics (Math-425)

Lecture Notes

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1 COUNTING

1.1 Counting Principles

There are two basic counting principles, the **product rule**, and the **sum rule**.

The Product Rule:

Suppose that a procedure can be broken into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to the procedure.

Example 1.1. *Let a man has 6 shirts and 5 pants, then he has $6 \times 5 = 30$ different ways to choose his shirt and pant.*

Exercise 1.2. *A new company with just two employees, Ahmed and Sameer rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees.*

Solution:

The procedure of assigning offices to these two employees consists of assigning an office to Ahmed, which can do in 12 ways, then assigning an office to Sameer different from the office to Ahmed, which can be done in 11 ways. By the product rule, there $12 \times 11 = 132$, ways to assign offices to these employees.

The Sum Rule:

If a task can be done either in one of n_1 ways or in one of n_2 ways, where non of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Exercise 1.3. *A student can choose project from one of the three lists. The three lists contain 23, 15 and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?*

Solution:

A student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

Definition 1.1. For an integer $n \geq 0$, n factorial, denoted by $n!$ is defined by $0! = 1$, and for $n \geq 1$,

$$n! = n(n-1)! \quad (1.1)$$

put $n = 1$ in (1.1), we get $1! = 1$.

Also, we can find $2! = 2 \cdot 1$, $3! = 3 \cdot 2 \cdot 1$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

In addition, for each $n \geq 0$, $(n+1)! = (n+1)(n!)$.

Also, we have:

$$n! = n(n-1)!$$

$$(n-1)! = (n-1)(n-2)!$$

$$(n-2)! = (n-2)(n-3)!$$

$$(n-r)! = (n-r)(n-r-1)!$$

$$(n-r)! = (n-r)(n-r-1)(n-r-2) \cdots (3)(2)(1)!$$

Exercise 1.4. Find $\frac{n!}{(n-2)!}$.

Solution: $\frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1)$.

Exercise 1.5. Find $\frac{n!}{(n-5)!}$ and $\frac{n!}{(n-r)!}$.

Exercise 1.6. *Is it true that for all $n, m \in \mathbb{Z}$, $(n + m)! = n! + m!$? Give an example.*

Also, $n! = m! \iff n = m$?

1.2 Permutations

Definition 1.2. A permutation of a set of distinct objects is an ordered arrangement of these objects.

Definition 1.3. An ordered arrangement of r -elements of a set is called an r -permutations. The number of r -permutations of a set with n elements is denoted by $P(n, r)$.

Exercise 1.7. *Let $S = \{1, 2, 3\}$. Find $P(3, 2)$ and $P(3, 3)$.*

Solution: First we will find the 2-permutation of the set S . These permutations are 12, 21, 13, 31, 23, 32. Therefore $P(3, 2) = 6$.
Next, we find $P(3, 3)$ these are 123, 132, 231, 213, 312. So, $P(3, 3) = 6$.

Theorem 1.8. *If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are*

$$P(n, r) = n(n - 1)(n - 2)(n - 3) \cdots (n - r + 1),$$

r -permutations of a set with n distinct elements.

Corollary 1.9. *If n and r are integers with $0 \leq r \leq n$, then*

$$P(n, r) = \frac{n!}{(n - r)!}. \text{ In particular } P(n, n) = n!, \text{ also } P(n, 0) = 1.$$

Proof. By Theorem 1.8

$$\begin{aligned}
 P(n, r) &= n(n-1)(n-2)(n-3) \cdots (n-r+1) \\
 &= n(n-1)(n-2)(n-3) \cdots (n-r+1) \frac{(n-r)(n-r-1) \cdots 3 \cdot 2 \cdot 1}{(n-r)(n-r-1) \cdots 3 \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}. \\
 P(n, n) &= \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!. \\
 P(n, 0) &= \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1. \quad \square
 \end{aligned}$$

Exercise 1.10. Find $P(5, 4)$, $P(5, 3)$, $P(5, 5)$ and $P(5, 0)$.

Exercise 1.11. How many ways are there to select a first prize winner, a second prize winner, and a third prize winner from 100 different people who have entered a contest.

Solution: $P(100, 3) = \frac{100!}{(100-3)!} = \frac{100!}{97!} = 100 \cdot 99 \cdot 98.$

Exercise 1.12. Find n if $P(n, 2) = 30$.

Solution: Since $P(n, r) = \frac{n!}{(n-r)!}$, then

$$\begin{aligned}
 \frac{n!}{(n-2)!} = 30 &\implies \frac{n(n-1)(n-2)!}{(n-2)!} = 30 \implies n(n-1) = 30 \\
 &\implies n^2 - n - 30 = 0 \implies (n-6)(n+5) = 30 \implies n = 6.
 \end{aligned}$$

1.3 Combinations

Definition 1.4. An r -combinations of a set is an unordered selection of r -elements from the set.

Thus, an r -combination is simply a subset of the set with r elements. The number of r -combinations of a set with n -distinct elements is denoted by $C(n, r)$ or $\binom{n}{r}$ and is called a binomial coefficient.

Exercise 1.13. Let $S = \{a, b, c, d\}$. Find all 3-combinations of the set S .

Solution: The 3-combinations of the set S are abc, abd, acd, bcd .

Exercise 1.14. Let $S = \{a, b, c, d\}$. Find all 4-combinations of the set S .

Solution: There is only one combination, namely $abcd$.

Now suppose we want to find all 2-permutations of S . First we find the 2-combination S namely ab, ac, ad, bc, bd, cd . Now reorder each combination we get ba, ca, da, cb, db, dc . So, $P(4, 2) = 2!C(4, 2)$.

Theorem 1.15. The number of r -combinations of a set with n -elements, where n is nonnegative integer and r is an integer with $0 \leq r \leq n$ is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Proof. The number of r , permutations of a set can be obtained by forming the $\binom{n}{r}$ r -combinations of the set, and then ordering the elements in each r -combinations which can be done in $P(r, r)$ ways. Then

$$P(n, r) = \binom{n}{r} P(r, r)$$

So,

$$\binom{n}{r} = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!} \cdot \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

□

Exercise 1.16. Find n if $\binom{n}{2} = 45$.

Solution: $\binom{n}{2} = 45 \implies \frac{n!}{2!(n-2)!} = 45 \implies \frac{n(n-1)(n-2)!}{2!(n-2)!} = 45$
 $\implies \frac{n(n-1)}{2 \cdot 1} = 45 \implies n^2 - n - 90 = 0 \implies (n-10)(n+9) = 0 \implies n = 10$.

Corollary 1.17. *Let n and r be nonnegative integers with $r \leq n$. Then*

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof. From Theorem 1.15, it follows that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Then

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

□

Corollary 1.18. *Let n, r_1 and r_2 be nonnegative integers. If $\binom{n}{r_1} = \binom{n}{r_2}$, then either $r_1 = r_2$ or $r_1 + r_2 = n$.*

Exercise 1.19. *Find r if $C(20, 2r) = C(20, r+2)$.*

Solution: By corollary 1.18, either $2r = r+2$ or $2r + r+2 = 20$. Solve these two equations, we get $r = 2$ or $r = 6$. The correct answer is $r = 6$.

Theorem 1.20. (*Pascal Identity*) Let n and r be positive integers with $n \geq r$.

Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Proof. Take right hand side:

$$\begin{aligned} \binom{n}{r-1} + \binom{n}{r} &= \frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!(n-r)!} \\ &= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} \\ &= \frac{n!}{(r-1)!(n-r+1)(n-r)!} + \frac{n!}{r(r-1)!(n-r)!} \\ &= \frac{n!}{(r-1)!(n-r)!} \left[\frac{1}{n-r+1} + \frac{1}{r} \right] \\ &= \frac{n!}{(r-1)!(n-r)!} \left[\frac{r+n-r+1}{r(n-r+1)} \right] \\ &= \frac{n!(n+1)}{(r-1)!(n-r)!r(n-r+1)} \\ &= \frac{(n+1)!}{r(r-1)!(n-r)!(n-r+1)} \\ &= \frac{(n+1)!}{r!(n+1-r)!} = \binom{n+1}{r} \end{aligned}$$

□

1.4 The Binomial Theorem

Theorem: Let x and y be variables and let n be nonnegative integer. Then

$$\begin{aligned}(x+y)^n &= \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots \\ &\quad + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n\end{aligned}$$

Proof. Proof by induction. For $n = 0$, we have

$$(x+y)^n = 1 = \binom{0}{0} x^0 y^0,$$

and hence the result is true. Assume the result is true for $n = k$, that is

$(x+y)^k = \sum_{j=0}^k \binom{k}{j} x^{k-j} y^j$. Now, we show that the result is true for $n = k+1$.

$$\begin{aligned}(x+y)^{k+1} &= (x+y)(x+y)^k = x(x+y)^k + y(x+y)^k \\ &= x \sum_{j=0}^k \binom{k}{j} x^{k-j} y^j + y \sum_{j=0}^k \binom{k}{j} x^{k-j} y^j \\ &= \sum_{j=0}^k \binom{k}{j} x^{k+1-j} y^j + \sum_{j=0}^k \binom{k}{j} x^{k-j} y^{j+1} \\ &= \binom{k}{0} x^{k+1} y^0 + \sum_{j=1}^k \binom{k}{j} x^{k+1-j} y^j + \sum_{j=1}^{k-1} \binom{k}{j} x^{k-j} y^{j+1} + y^{k+1} \quad \square \\ &= x^{k+1} + \sum_{j=1}^k \binom{k}{j} x^{k+1-j} y^j + \sum_{j=1}^k \binom{k}{j-1} x^{k+1-j} y^j + y^{k+1} \\ &= x^{k+1} + \sum_{j=1}^k \left[\binom{k}{j} + \binom{k}{j-1} \right] x^{k+1-j} y^j + y^{k+1} \\ &= \sum_{j=0}^{k+1} \binom{k+1}{j} x^{k+1-j} y^j\end{aligned}$$

Exercise 1.21. *What is the expansion of $(x + y)^4$?*

Solution:

$$\begin{aligned}(x + y)^4 &= \sum_{j=0}^4 \binom{4}{j} x^{4-j} y^j \\&= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \\&= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4.\end{aligned}$$

Exercise 1.22. *Find the coefficients of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$.*

Solution: By the binomial theorem

$$\binom{n}{j} x^{n-j} y^j = \binom{25}{13} x^{12} y^{13}.$$

Therefore, the coefficients of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$ is

$$\binom{25}{13} = 5200300.$$

Exercise 1.23. *What are the coefficients of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?*

Solution: By the binomial theorem

$$(2x - 3y)^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j.$$

Therefore, coefficients of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$ is

$$\binom{25}{13} (2)^{12} (-3)^{13}.$$

Corollary 1.24. *Let n be a nonnegative integer. Then $\sum_{k=0}^n \binom{n}{k} = 2^n$.*

Proof. Using the binomial theorem with $x = 1$ and $y = 1$, we get

$$(1 + 1)^n = 2^n = \sum_{k=0}^n \binom{n}{k} 1^n 1^{n-k} = \sum_{k=0}^n \binom{n}{k}.$$

□

Corollary 1.25. *Let n be a positive integer. Then $\sum_{k=0}^n \binom{n}{k} (-1)^k = 0$.*

Proof. Put $x = 1$ and $y = -1$ in the binomial Theorem

We get $(1 + (-1))^n = 0 = \sum_{k=0}^n \binom{n}{k} (1)^{n-k} (-1)^k = \sum_{k=0}^n \binom{n}{k} (-1)^k$.

Expanding the sum, we get

$$\binom{n}{0} (-1)^0 + \binom{n}{1} (-1)^1 + \binom{n}{2} (-1)^2 + \binom{n}{3} (-1)^3 + \cdots = 0$$

Or

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0$$

Or

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

□

Corollary 1.26. *Let n be a nonnegative integer. Then $\sum_{k=0}^n \binom{n}{k} 2^k = 3^n$.*

Proof. By binomial Theorem with $x = 1$ and $y = 2$, we have

$$(1 + 2)^n = 3^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^n \binom{n}{k} 2^k.$$

□

Theorem 1.27. *Let n, m and r be nonnegative integers and r not exceeding either n or m . Then, $\binom{n+m}{r} = \sum_{k=0}^r \binom{n}{r-k} \binom{m}{k}$.*

Proof. Suppose that there are n items in one set and m items in a second set. Then the total number of ways to pick r items from the union of these sets is $\binom{n+m}{r}$.

Another way to pick r items from the union is to pick k items from the first set and then $r - k$ items from the second set, $0 \leq k \leq r$. This can be done in $\binom{n}{k} \binom{m}{r-k}$ ways, using the product rule, we get $\binom{n+m}{r} = \sum_{k=0}^r \binom{n}{k} \binom{m}{r-k}$. $\square$

Corollary 1.28. *If n is a nonnegative integer, then*

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2.$$

Proof. Put $n = m = r$ in Theorem 1.27, we get

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{n-k}$$

because $\binom{n}{k} = \binom{n}{n-k}$. $\square$

Exercise 1.29. *Show that if n and k are positive integers, then*

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Solution: Since

$$\binom{n+m}{r} = \sum_{k=0}^r \binom{n}{r-k} \binom{m}{k}.$$

Then

$$\begin{aligned}
 \binom{2n}{2} &= \sum_{k=0}^2 \binom{n}{2-k} \binom{n}{k} \\
 &= \binom{n}{2-0} \binom{n}{0} + \binom{n}{2-1} \binom{n}{1} + \binom{n}{2-2} \binom{n}{2} \\
 &= \binom{n}{2} + \binom{n}{1} \binom{n}{1} + \binom{n}{0} \binom{n}{2} = 2 \binom{n}{2} + n^2.
 \end{aligned}$$

1.4.1 term in the Binomial expansion

The general term in the Binomial expansion is given by $\mathbf{T}_{j+1} = \binom{n}{j} x^{n-j} y^j$.

Exercise 1.30. Find the seventh term in the expansion of $(2x - y)^9$.

Solution: Since $\mathbf{T}_{j+1} = \binom{n}{j} x^{n-j} y^j$, then

$$\mathbf{T}_7 = \mathbf{T}_{6+1} = \binom{9}{6} (2x)^{9-6} (-y)^6 = 672x^3y^6.$$

Exercise 1.31. Find the coefficients of x^{10} in the expansion of $(2x - 3)^{14}$.

Solution: Since the order of the term is unknown, we assume that the term contains x^{10} is $\mathbf{T}_{j+1}$.

Therefore,

$$\mathbf{T}_{j+1} = \binom{14}{j} (2x)^{14-j} (-3)^j.$$

To get the term contains x^{10} , put $x^{10} = x^{14-j}$. Then $14 - j = 10$, implies $j = 4$. So, $\mathbf{T}_5 = \binom{14}{4} (2x)^{14-4} (-3)^4$.

Therefore, the coefficient of x^{10} in the expansion of $(2x-3)^{14}$ is $\binom{14}{4} (2)^{10} (-3)^4$.

Exercise 1.32. Find the term which is free from the variable x in the expansion of $(x - x^{-1})^{10}$.

Solution: $\mathbf{T}_{j+1} = \binom{n}{j} x^{n-j} y^j = \binom{10}{j} x^{10-j} (x^{-1})^j = \binom{10}{j} x^{10-j} x^{-j} = \binom{10}{j} x^{10-j-j}$. Put $10 - j - j = 0$, implies $j = 5$. Therefore, the term free from x is T_6 .

1.4.2 The ratio between two terms

Ratio between two consecutive terms in $(x + y)^n$ is

$$\frac{T_{j+1}}{T_j} = \frac{\binom{n}{j} x^{n-j} y^j}{\binom{n}{j-1} x^{n-j+1} y^{j-1}} = \frac{(n-j+1)y}{jx}.$$

Exercise 1.33. Find the ratio between the fifth term and fourth term in $(3x + 4y)^{12}$.

Solution: Since $\frac{T_{j+1}}{T_j} = \frac{(n-j+1)y}{jx}$, then

$$\frac{T_5}{T_4} = \frac{(12-4+1)4y}{4 \cdot 3x} = \frac{3y}{x}.$$

1.4.3 The ratio between two coefficients of terms

In the expansion of $(ax+by)^n$, $(\text{coefficient of } T_{j+1} / \text{coefficient } T_j) = \frac{n-j+1}{j} \times \frac{b}{a}$.

Exercise 1.34. Find n and x if the coefficients of the 8th term and 9th in $(1+x)^n$ are equal and given that $\frac{T_7}{T_5} = \frac{44}{75}$.

Solution: $\frac{\text{Coff } T_9}{\text{Coff } T_8} = \frac{(n-8+1)}{8} \times \frac{1}{1} = 1$. Then, $n - 7 = 8 \Rightarrow n = 15$.

$$\begin{aligned} \frac{T_7}{T_5} &= \frac{T_7}{T_6} \cdot \frac{T_6}{T_5} = \frac{44}{75} \\ \left(\frac{15-6+1}{6} \cdot \frac{x}{1} \right) \cdot \left(\frac{15-5+1}{5} \cdot \frac{x}{1} \right) &= \frac{44}{75} \\ \Rightarrow x^2 &= \frac{4}{25} \Rightarrow x = \pm \frac{2}{5}. \end{aligned}$$

1.4.4 Middle terms in $(x + y)^n$

Since the number of terms $(x + y)^n$ is $n + 1$, then we have two cases:

1. If n is even, then $n + 1$ is odd and in this case, we have only one middle term, its order is $\frac{n}{2} + 1$.

2. If n is odd, then $n + 1$ is even, and we have two middle terms, the first is of order $\frac{n+1}{2}$ and the second has order $\frac{n+1}{2} + 1$.

Exercise 1.35. Find the middle terms in the expansion of $(x - y)^9$.

Solution: Order of the first term is $\frac{9+1}{2} = 5$ and the order of the second is 6. Since $T_{j+1} = \binom{n}{j}x^{n-j}y^j$, then

$$T_5 = \binom{9}{4}x^{9-4}(-y)^4 = \binom{9}{4}x^5y^4$$

and

$$T_6 = \binom{9}{5}x^{9-5}(-y)^5 = -\binom{9}{5}x^4y^5.$$

Exercises

- List all the permutations of the set $\{a, b, s\}$.
- Let $S = \{1, 2, 3, 4, 5\}$.
 - List all the 3-permutations of S .
 - List all the 3-combinations of S .
- Find the value of these quantities: $P(6, 3), P(6, 4), P(8, 1), P(8, 8)$.
- Find the values of: $C(8, 0), C(5, 3), C(8, 8), C(12, 6)$.
- Find the expansion of $(x + y)^6$.
- What is the coefficient of x^9 in $(2 - x)^{19}$.
- What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$.
- Find the 8th term in the expansion $(2x^3 + \frac{1}{x})^{10}$.
- Find the coefficient of the seventh term in $(3x^2 - \frac{2x}{3})^{11}$.
- Find the middle terms in the expansion of $(x - y)^{11}$.

11. Prove that if n and k are integers with $1 \leq k \leq n$, then

$$k \binom{n}{k} = n \binom{n-1}{k-1}.$$

12. Prove that

$$\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}.$$

13. Show that if n and k are positive integers, then

$$\binom{2n}{2} = 2 \binom{n}{2} + n^2.$$

14. Show that if n and k are positive integers, then

$$\binom{n+1}{k} = \frac{(n+1) \binom{n}{k-1}}{k}$$

15. Show that if n is an integer then,

$$\sum_{k=0}^n 3^k \binom{n}{k} = 4^n.$$

16. Find n if (a) $P(n, 2) = 110$, (b) $P(n, n) = 5040$.

17. Find n if $C(n, 2) = 45$.

18. Find r if $C(28, r+4) = C(28, r-2)$

19. Show that if n and r are nonnegative integers and $n \geq r$, then

$$P(n+1, r) = \frac{P(n, r)(n+1)}{(n+1-r)}.$$

2 GRAPHS

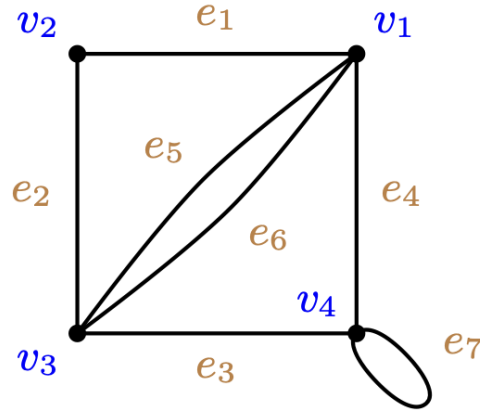
2.1 Basic Definitions

Definition 2.1. A graph G is a triple (V, E, I) where

- V (or $V(G)$) is a finite set whose elements are called vertices;
- E (or $E(G)$) is a finite set disjoint from V whose elements are called edges; and
- I , called the incidence relation, is a subset of $V \times E$ in which each edge is in relation with exactly one or two vertices.

Example 2.1. Let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and I is given by

$$I = \{(v_1, e_1), (v_1, e_4), (v_1, e_5), (v_1, e_6), (v_2, e_1), (v_2, e_2), (v_3, e_2), (v_3, e_3), (v_3, e_5), (v_3, e_6), (v_4, e_3), (v_4, e_4), (v_4, e_7)\}.$$

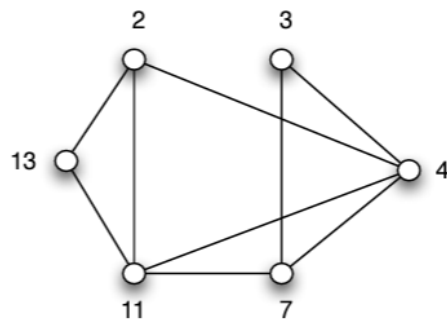


Exercise 2.2. Let $S = \{2, 3, 4, 7, 11, 13\}$. Draw the graph G whose vertex set is S and such that (i, j) is an edge of G for $i, j \in S$ if $i + j \in S$ or $|i - j| \in S$.

Solution: In the following table, we show all values of $i + j$ and $|i - j|$ for $i, j \in S$, where the value written by RED font indicates that the value is in S .

(i, j)	$i + j$	$ i - j $
(2, 3)	5	1
(2, 4)	6	2
(2, 7)	9	5
(2, 11)	13	9
(2, 13)	15	11
(3, 4)	7	1
(3, 7)	10	4
(3, 11)	14	8
(3, 13)	16	10
(4, 7)	11	3
(4, 11)	15	7
(4, 13)	17	9
(7, 11)	18	4
(7, 13)	20	6
(11, 13)	24	2

From this table, we can see that G contains 6 vertices and 9 edges, which is as shown in the following figure.



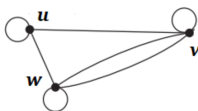
Definition 2.2. If $e = uv$, then the vertices u and v are the end vertices of the edge e , we say, e is incident with vertices u and v .



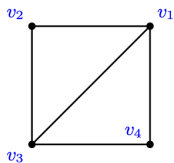
Definition 2.3. A set of two or more edges of a graph G is called a set of multiple edges or parallel edges if they have the same ends.



Definition 2.4. An edge for which the two ends are the same is called a loop at the common vertex.



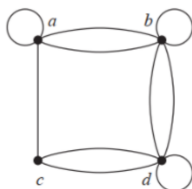
Definition 2.5. Graphs with no parallel edges and no loops are called simple.



Definition 2.6. A graph G is called a multi graph if it has no loops. Any graph which contains some multiple edges is called a multi graph. In a multi graph, no loops are allowed.



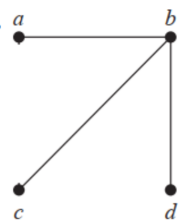
Definition 2.7. A graph G is called a pseudo if it has loops and parallel edges.



Definition 2.8. Two vertices u and v are adjacent if and only if there is an edge e start with u and ends with v .



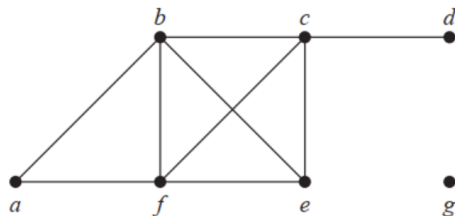
Example 2.3. Vertex a is adjacent to vertex b , vertex b is adjacent to the vertices c and d but vertices a , c , d are nonadjacent to each other, also we can say vertices a , c , d are adjacent to the vertex b .



Definition 2.9. The degree of a vertex in a graph G is the number of edges incident with it, and is denoted by $\deg(v)$. A vertex of degree zero is called an isolated vertex and a vertex of degree one is called pendant vertex.

Definition 2.10. A vertex in a graph G is called odd or even depending on whether its degree is odd or even.

Exercise 2.4. Find the degree of each vertex of the following graph.



Solution:

$$\deg(a) = 2$$

$$\deg(b) = 4$$

$$\deg(c) = 4$$

$$\deg(d) = 1$$

$$\deg(e) = 3$$

$$\deg(f) = 4$$

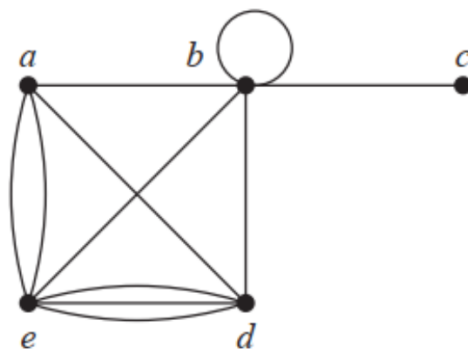
and

$$\deg(g) = 0.$$

So, in the given graph the vertex g is an isolated vertex while the vertex d is a pendant vertex.

Note: If there is a loop, we count it twice.

Exercise 2.5. Find the degree of each vertex of the following graph.



Solution:

$$\deg(a) = 4$$

$$\deg(b) = 6$$

$$\deg(c) = 1$$

$$\deg(d) = 5$$

and

$$\deg(e) = 6.$$

2.2 Order and size of a graph

Definition 2.11. Let $G = (V, E)$. Then, the order of G , denoted by $|V(G)|$ or $|V|$, is the number of vertices and the size of G , denoted by $|E(G)|$ or $|E|$ is the number of edges. So, if $|V| = n$ $|E| = q$, then the order of G is n and is size is q and we called such a graph is a (n, q) graph.

So, the order and size of the graph given in the previous example are 5 and 11 respectively.

The first theorem of graph theory was due to Leonhard Euler (1707-1783). This theorem connects the degrees of the vertices and the number of edges.

Theorem 2.6. (*The hand shaking theorem*) Let G be a graph of order n and size q . Then

$$\sum_{v \in G} \deg(v) = 2q.$$

Proof. Each edge, since it has two ends' vertices contributes precisely 2 to the sum of the degrees, i.e., when the sum of the vertices are summed, each edge is counted twice. $\square$

Exercise 2.7. *How many vertices are there in a graph with 30 edges such that each vertex is of degree 6?*

Solution: Let n be the number of the vertices of such a graph. By the hand shaking theorem, we have

$$6n = 2(30) \implies n = 10.$$

Exercise 2.8. *Let G be a graph of order 10 and size 30, such that all its vertices have the same degree. Find the degree of each vertex.*

Solution: Since all the vertices have the same degree, let r be the degree of each vertex. Therefore

$$10r = 2(30) \implies r = 6.$$

Corollary 2.9. *If G is a graph of order n and size q , each vertex of G is of degree r . Then, $nr = 2q$.*

Exercise 2.10. *A certain graph G has order 14 and size 27. The degree of each vertex of G is 3, 4 or 5. There are 6 vertices of degree 4. How many vertices of G have degree 3 and how many have degree 5?*

Solution: Let x be the number of vertices of G of degree 3, since order of G is 14 and 6 vertices of degree 4, then there are $(8 - x)$ vertices of degree 5. Therefore,

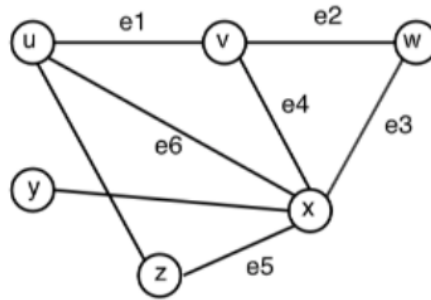
$$(4 \times 6) + 3x + 5(8 - x) = (2 \times 27) \implies 24 + 3x + 40 - 5x = 54 \implies x = 5.$$

So, there are 5 vertices of degree 3 and there are 3 vertices of degree 5.

Corollary 2.11. *In any graph G , there is an even number of odd vertices.*

Definition 2.12. The maximum degree of a vertex of a graph G is denoted by $\Delta(G)$ and the minimum is denoted by $\delta(G)$.

Exercise 2.12. Find the maximum and minimum degree for the following graph.



Solution: $\Delta(G) = 5$ and $\delta(G) = 1$.

Theorem 2.13. For any simple graph G of order n , $\Delta(G) \leq n - 1$.

Proof. Let $v \in V(G)$. Since G is simple, then no multiple edges nor loops are allowed. Then v can be adjacent to at most $n - 1$ vertices in G . Thus, $\Delta(G) \leq n - 1$. $\square$

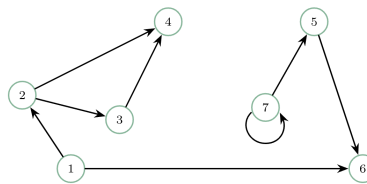
Corollary 2.14. For any simple graph G of order n ,

$$0 \leq \delta(G) \leq \Delta(G) \leq n - 1.$$

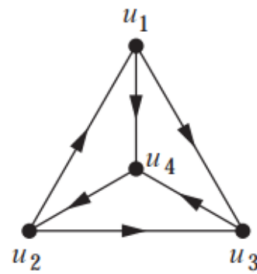
Theorem 2.15. Let G be a simple graph of order n and size q . Let $\delta(G)$ and $\Delta(G)$ be the minimum degree and maximum degree of G , respectively. Then, $\delta(G) \leq \frac{2q}{n} \leq \Delta(G)$.

2.3 Directed Graphs

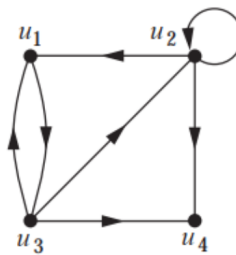
Definition 2.13. A directed graph (or digraph) $G = (V, E)$ consists of a nonempty set of vertices V and a set of direct edges E . Each edge is associated with an ordered pair of vertices.



Definition 2.14. When a directed graph has no loops and has no multiple edges it is called a simple directed graph.



Definition 2.15. When a directed graph has multiple edges, it is called the directed multi graph.

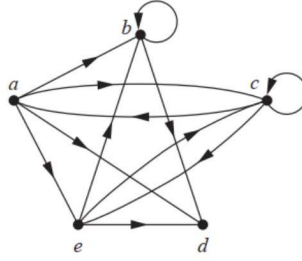


2.3.1 Degree of a vertex in a directed graph

Definition 2.16. The in-degree of a vertex v , denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The out-degree of a vertex v , denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex.

Note that: a loop at a vertex contributes one to both the in-degree and the out-degree of this vertex.

Exercise 2.16. Find the in-degree and the out-degree of the following digraph.



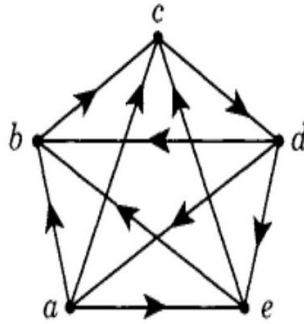
Solution: $\deg^-(a) = 1, \deg^-(b) = 3, \deg^-(c) = 3, \deg^-(d) = 3, \deg^-(e) = 2$.
 $\deg^+(a) = 4, \deg^+(b) = 2, \deg^+(c) = 3, \deg^+(d) = 0, \deg^+(e) = 3$.

Theorem 2.17. Let $G = (V, E)$ be a directed graph. Then,

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

Proof. Since $\deg^-(v_i)$ denotes the number of edges terminating at v_i , each is counted exactly once in $\sum_{i=1}^n \deg^-(v_i) = |E|$. Similarly, $\sum_{i=1}^n \deg^+(v_i) = |E|$. □

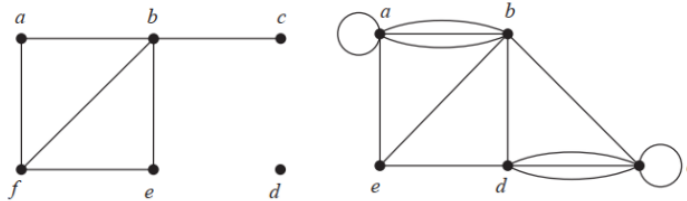
Exercise 2.18. Find the in-degree and the out-degree of the following digraph.



Solution: $\deg^-(a) = 1, \deg^-(b) = 3, \deg^-(c) = 3, \deg^-(d) = 1, \deg^-(e) = 2.$
 $\deg^+(a) = 3, \deg^+(b) = 1, \deg^+(c) = 1, \deg^+(d) = 3, \deg^+(e) = 2.$

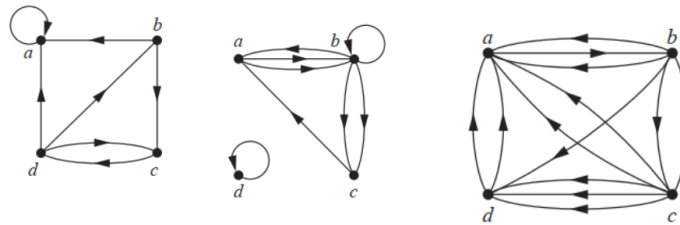
Exercises

1. Let $S = \{-6, -3, 0, 3, 6\}$. Draw the graph G whose vertex set is S and such that $ij \in E(G)$, for $i, j \in S$ if $i + j \in S$ or $|i - j| \in S$.
2. Find the number of vertices, the number of edges, and the degree of each vertex in the given graphs. Identify all isolated and pendent vertices.



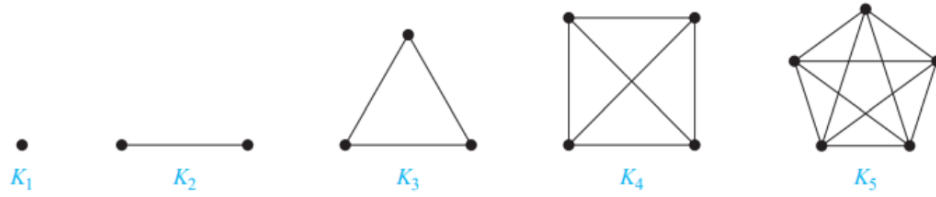
3. Find the sum of the degrees of the vertices of each graph in Exercises 2, and verify that it equals twice the number of edges in the graph.
4. Can a simple graph exist with 15 vertices each of degree five?

5. Give an example of the following or explain why no such example exists:
 - (a) a graph of order 7 whose vertices have degrees 1,1,1,2,2,3,3.
 - (b) a graph of order 7 whose vertices have degrees 1,2,2,2,3,3,7
 - (c) a graph of order 4 whose vertices have degrees 1,3,3,3.
6. The degree of each vertex of a certain graph of order 12 and size 31 is (either 4 or 6). How many vertices of degree 4 are there?
7. Show that the max. number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.
8. Let G be a graph with n vertices and q edges. Let m be the smallest positive integer such that $m \geq \frac{2q}{n}$. Prove that G has a vertex of degree at least m .
9. Determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multi graph.



2.4 Some special graphs

Definition 2.17. (Complete graphs) The complete graph of order n , denoted by K_n is the simple graph that contains exactly one edge between each pair of distinct vertices. In K_n , every vertex has degree $n - 1$.



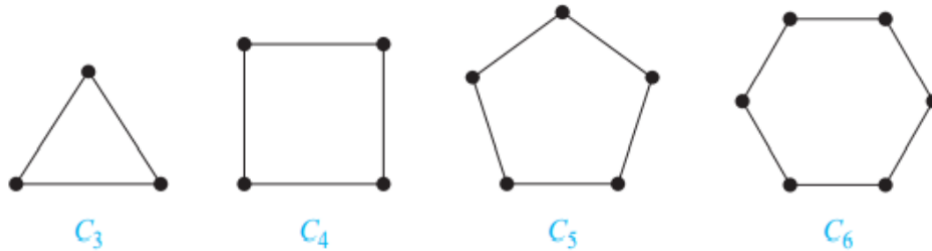
Theorem 2.19. The number of edges of K_n is $\binom{n}{2}$.

Proof. Since the number of vertices of K_n is n and for all $v \in V(K_n)$, $\deg(v) = n - 1$, then by the hand shaking theorem, we have

$$n(n - 1) = 2q \implies q = \frac{n(n - 1)}{2} = \binom{n}{2}.$$

□

Definition 2.18. (Cycle graphs) The cycle graph $C_n, n \geq 3$ consists of n vertices $v_1, v_2, \dots, v_n$ and edges $v_1v_2, v_2v_3, v_3v_4, \dots, v_{n-1}v_n, v_nv_1$. The number of edges of the cycle $C_n, n \geq 3$, is n edges.

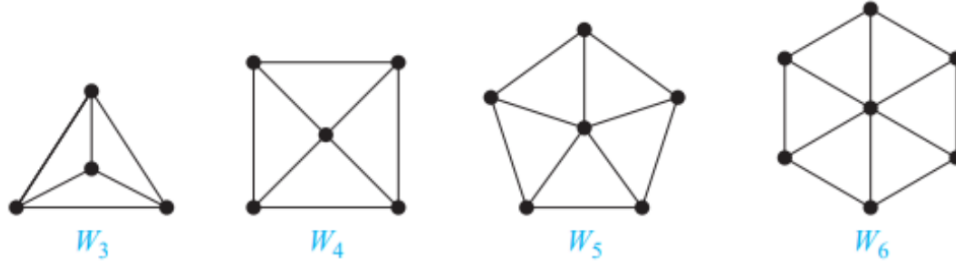


Theorem 2.20. $|E(C_n)| = n$.

Proof. The number of vertices of C_n is n and each vertex is of degree 2. Then by the hand shaking theorem, theorem follows. $\square$

Definition 2.19. A cycle graph $C_n, n \geq 3$ is called an odd cycle if its size is odd and is called even if its size is even. For example, C_3 is an odd cycle while C_4 is an even cycle.

Definition 2.20. (Wheel graphs) We obtain the wheel graph W_n , when we add an additional vertex to the cycle graph C_n , for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n , by new edges.



Theorem 2.21. $|E(W_n)| = 2n$.

Proof. The number of vertices of W_n is $n + 1$ and there are n vertices is of degree 3 and one vertex is of degree n . Then, by the hand shaking theorem

$$3n + 1n = 2q \implies q = 2n.$$

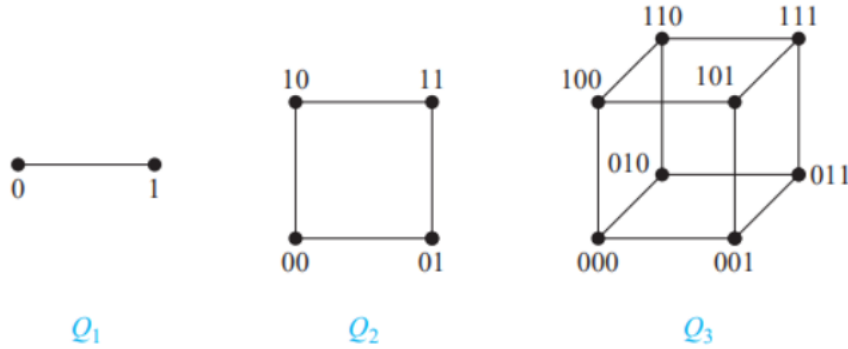
$\square$

Definition 2.21. (n-Cube graphs) The n -cube graph, denoted by Q_n , is the graph has vertices representing the 2^n bit strings of length n . Two vertices are adjacent if and only if the bit strings that are represent differ in exactly one-bit position.

For $n = 1$, then Q_1 has 2^1 vertices, namely 0 and 1.

For $n = 2$, then Q_2 has 2^2 vertices, namely 00, 01, 10, 11.

For $n = 3$, then Q_3 has 2^3 vertices, namely 000, 001, 010, 011, 100, 101, 110, 111.



Theorem 2.22. $|E(Q_n)| = n2^{n-1}$.

Proof. The number of vertices of Q_n is 2^n and each vertex is of degree n .

Then by the hand shaking theorem, we have

$$n2^n = 2q \implies q = n2^{n-1}.$$

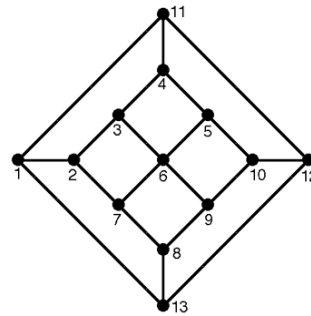
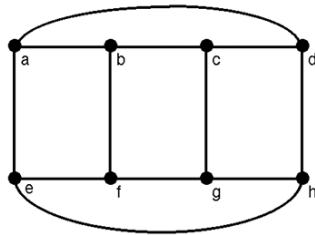
□

Definition 2.22. (Regular graphs) A graph G is regular if all its vertices have the same degrees. For example, K_n and C_n are regular graphs.

2.5 Bipartite Graphs and Complete Bipartite Graphs

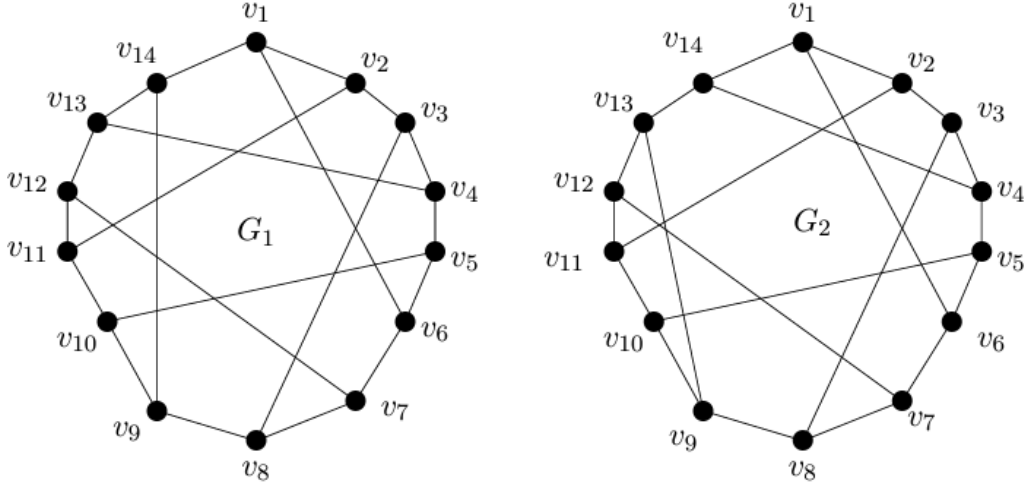
Definition 2.23. (Bipartite graph) A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G .

Exercise 2.23. Determine whether the following graphs are bipartite. If so, give the partition of the vertices into two sets.



Solution: The first one is (take the sets $\{a, f, c, h\}$ and $\{b, e, g, d\}$ as the two sets) and the second one isn't. The second one can't be because if it is bipartite then the partition of vertices will contain 1 in one set, 2 will be a second set, 7 in the first set, 8 must be in the second set, 13 must be in the first set and this leads to a contradiction since there is an edge between 13 and 1 and they are in the same bipartite set.

Exercise 2.24. Determine whether each of the following graphs G_1 and G_2 are bipartite. Justify your answers.

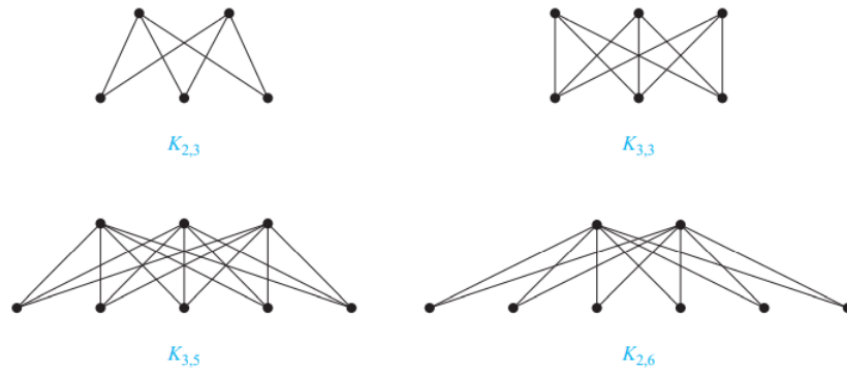


Solution: G_1 is bipartite since $X = \{v_1, v_3, v_5, v_7, v_9, v_{11}, v_{13}\}$ and $Y = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}\}$ is a bipartition of G . (It is easy to check that every edge of G has one end in X and one end in Y .)

G_2 is not bipartite because if G_2 has bipartition $\{X, Y\}$ then we may assume $v_1 \in X$. This implies that: $v_2, v_6, v_{14} \in Y$; $v_3, v_{11} \in X$; $v_4, v_8 \in Y$ and this leads to a contradiction since there is an edge between v_4 and v_{14} and they are in the same bipartite set Y .

Definition 2.24. (Complete bipartite graph) A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$ and $K_{2,6}$ are displayed in below Figure.



Theorem 2.25. $|E(K_{m,n})| = mn$.

Proof. The number of vertices of $K_{m,n}$ is $m+n$ and there are m vertices each of degree n and there are n vertices each of degree m . Then, by the hand shaking theorem, we have

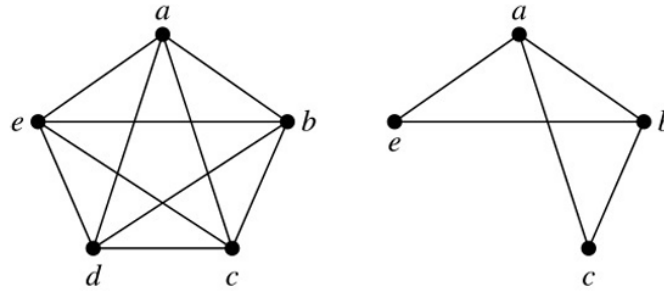
$$mn + nm = 2q \implies q = mn.$$

□

2.6 New Graphs from Old

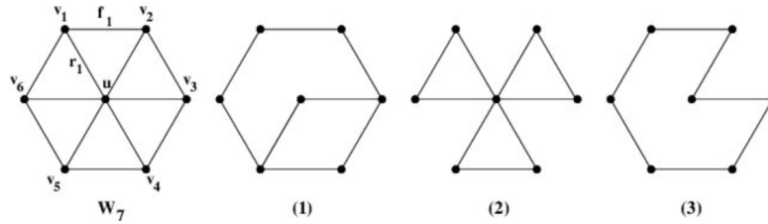
Definition 2.25. (Subgraph) A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subset V$ and $F \subset E$. A subgraph H of G is a proper subgraph of G if $H \neq G$.

Example 2.26. Here we show K_5 and one of its subgraphs.



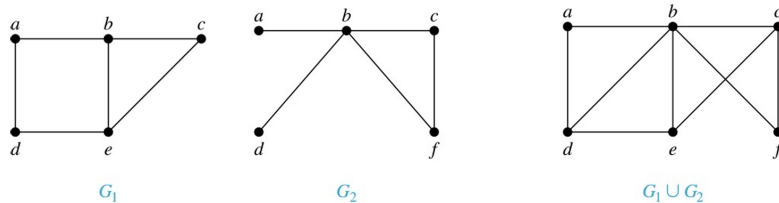
Definition 2.26. (Spanning subgraph) If a subgraph of a graph G has the same vertex set as G , then it is called a spanning subgraph of G .

For example,



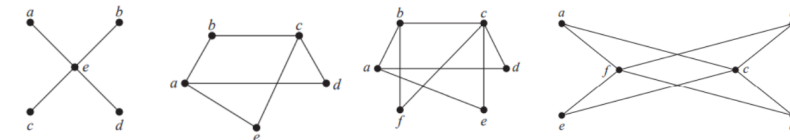
Definition 2.27. (Union of two Simple graphs) The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

For example,

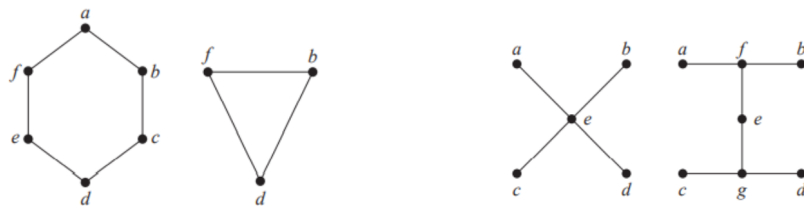


Exercises

- 1 Draw the following graphs: K_7 , C_7 , $K_{1,8}$, $K_{4,4}$, W_7 , Q_4 .
- 2 Determine which of the following graphs are bipartite:



- 3 How many subgraphs with at least one vertex does K_2 have?
- 4 Find the union of the following pair of simple graphs:



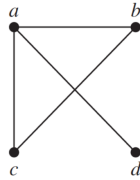
2.7 Graphs and Matrices

Definition 2.28. (Adjacency Matrices) Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \dots, v_n$. The adjacency matrix A (or A_G) of G , with respect to this listing of the vertices, is then $n \times n$ zero-one matrix with 1 as

its (i, j) th entry when v_i and v_j are adjacent, and 0 as its (i, j) th entry when they are not adjacent. In other words, if its adjacency matrix is $A = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G \\ 0 & \text{if } (v_i, v_j) \text{ otherwise.} \end{cases}$$

Exercise 2.27. Use an adjacency matrix to represent the given graph.



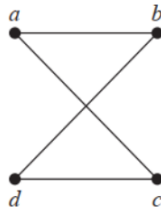
Solution:

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 2.28. Draw the graph with the adjacency matrix.

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

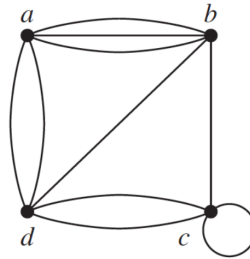
Solution:



Note: If G is a simple graph, then the adjacency matrix has the following properties:

- 1- The diagonal entries are all 0.
- 2- The adjacency matrix is symmetric.
- 3- $\deg(v_i)$ is the sum of the entries in the row i (or column j).

Exercise 2.29. Use an adjacency matrix to represent the given graph.



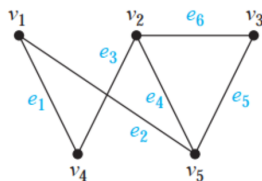
Solution:

$$A_G = \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Definition 2.29. (Incidence Matrices) Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \dots, v_n$ are the vertices and $e_1, e_2, \dots, e_m$ are the edges of G . Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2.30. Use an incident matrix to represent the given graph.

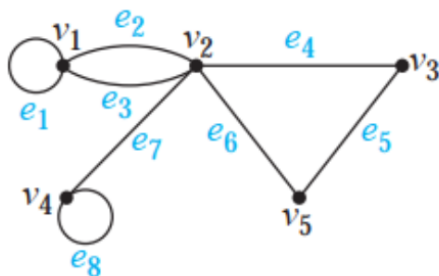


Solution: $M = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$

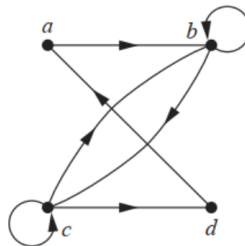
Exercise 2.31. Draw the graph with the following incident matrix.

$$M = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

Solution:



Exercise 2.32. Use an adjacency matrix to represent the given directed graph.



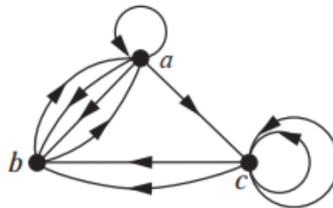
Solution:

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 2.33. Draw the graph represented by the following matrix.

$$A_G = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

Solution:

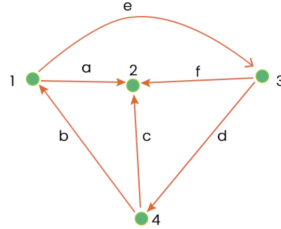


Definition 2.30. (Incidence Matrix for direct graph) Let $G = (V, E)$ be a directed graph with no loops. Suppose that $v_1, v_2, \dots, v_n$ are the vertices

and $e_1, e_2, \dots, e_m$ are the edges of G . Then the incidence matrix is the $n \times m$ matrix $B = [b_{ij}]$, where

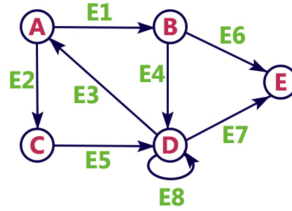
$$b_{ij} = \begin{cases} 1 & \text{If the edge } e_j \text{ is directed away from the vertex } v_i \\ -1 & \text{If the edge } e_j \text{ is directed towards the vertex } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2.34. Use an incident matrix to represent the given graph.



Solution: $M = \begin{matrix} & \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{pmatrix} \end{matrix}$

Exercise 2.35. Use an incident matrix to represent the given graph.

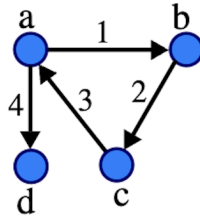


Solution: $M = \begin{matrix} & \begin{matrix} E1 & E2 & E3 & E4 & E5 & E6 & E7 & E8 \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix} \end{matrix}$

Exercise 2.36. Draw the graph with the following incident matrix.

$$M = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{matrix}$$

Solution:

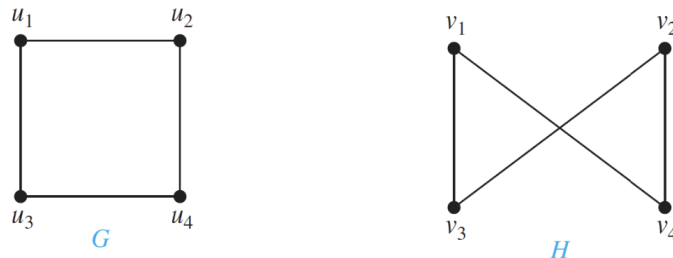


2.7.1 Isomorphism of Graphs

Definition 2.31. (Isomorphism of Graphs)

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one to one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2 , for all a and b in V_1 . Such a function f is called an isomorphism. Two simple graphs that are not isomorphic are called nonisomorphic.

Exercise 2.37. Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure, are isomorphic.

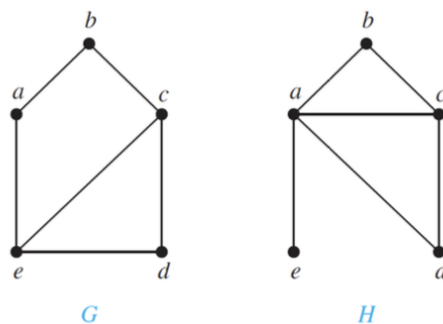


Solution: If we take $f : V \longrightarrow W$ such that $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$ and $f(u_4) = v_2$.

For instance,

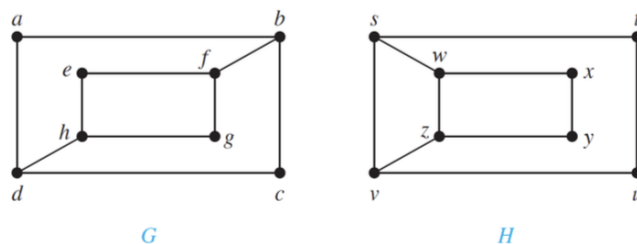
- 1 Isomorphic simple graphs must have the same number of vertices, because there is a one-to-one correspondence between the sets of vertices of the graphs.
- 2 Isomorphic simple graphs also must have the same number of edges, because the one-to-one correspondence between vertices establishes a one-to-one correspondence between edges.
- 3 The degrees of the vertices in isomorphic simple graphs must be the same. That is, a vertex v of degree d in G must correspond to a vertex $f(v)$ of degree d in H , because a vertex w in G is adjacent to v if and only if $f(v)$ and $f(w)$ are adjacent in H .

Exercise 2.38. Show that the graphs displayed in Figure are not isomorphic.



Solution: Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e , whereas G has no vertices of degree one. It follows that G and H are not isomorphic.

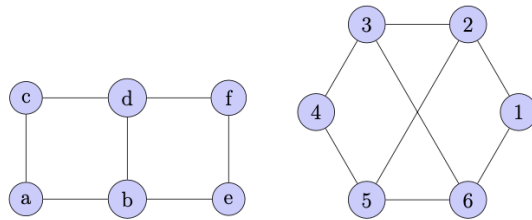
Exercise 2.39. Determine whether the graphs shown in Figure are isomorphic.



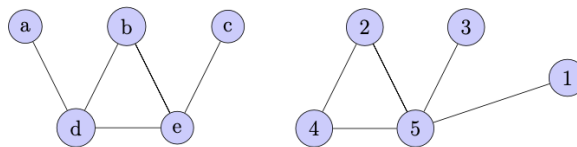
Solution: G and H are not isomorphic. To see this, note that because $\deg(a) = 2$ in G , a must correspond to either t, u, x , or y in H , because these are the vertices of degree two in H . However, each of these four vertices in H is adjacent to another vertex of degree two in H , which is not true for a in G . It follows that G and H are not isomorphic.

Exercise 2.40. Determine whether the following pairs of graphs are isomorphic. If not determine why.

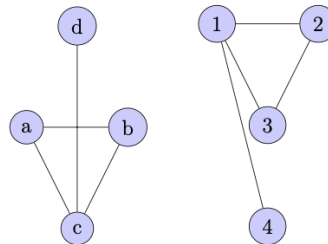
(a)



(b)



(c)

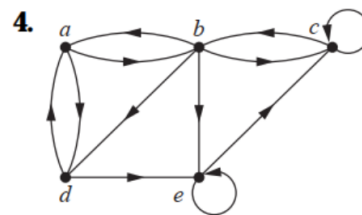
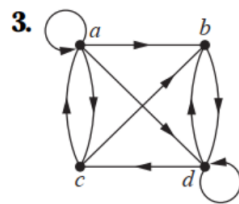
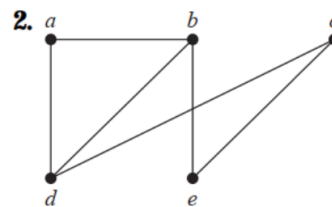
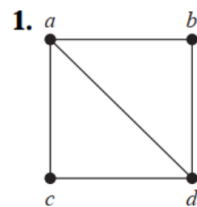


Solution:

- (a) Non-isomorphic. The degree sequences differ. The graph on the left has two vertices of degree 3, while the graph on the right has four vertices of degree 3.
- (b) Non-isomorphic. The graph on the right contains a vertex of degree 4, while the graph on the left does not.
- (c) Isomorphic.

Exercises

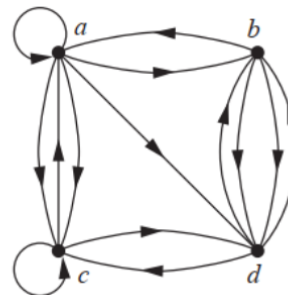
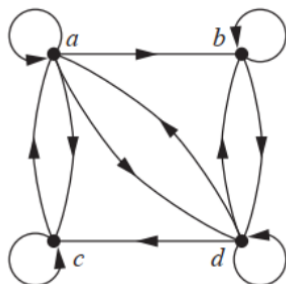
- 1 Represent each of these graphs with an adjacency matrix: K_4 , $K_{1,4}$, $K_{2,3}$, C_4 , W_4 and Q_3 .
- 2 Use an adjacency matrix to represent the given graph.



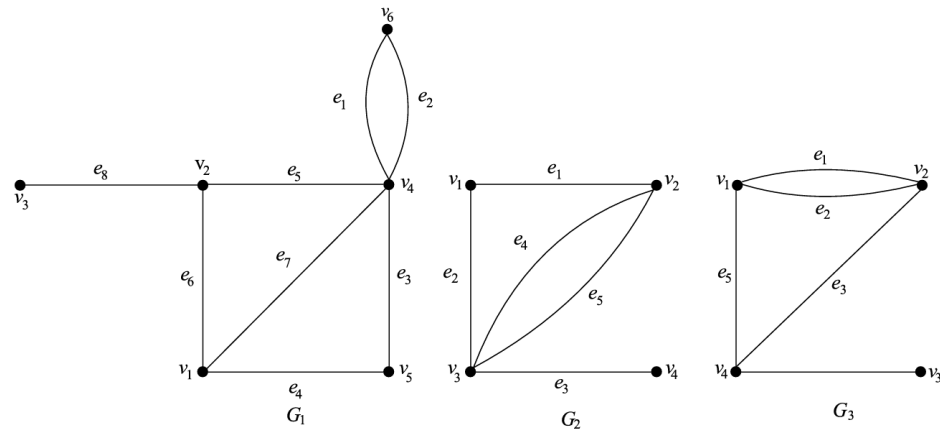
- 3 Draw a graph with the given adjacency matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

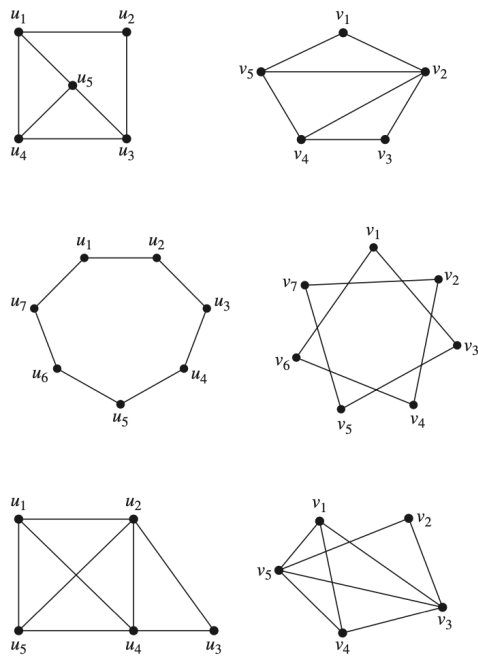
- 4 Find the adjacency matrix of the given directed multigraphs.



5 Use an incidence matrix to represent the given graph.



6 Determine whether the following pairs of graphs are isomorphic. If not determine why.



2.8 Connectivity

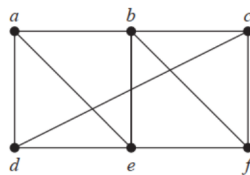
Definition 2.32. Let n be a nonnegative integer and G an undirected graph. A path of length n from u to v in G is a sequence of n edges $e_1, \dots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \dots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, \dots, n$, the endpoints x_{i-1} and x_i . When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \dots, x_n$ (because listing these vertices uniquely determines the path). The path is a circuit if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero. The path or circuit is said to pass through the vertices $x_1, x_2, \dots, x_{n-1}$ or traverse the edges $e_1, e_2, \dots, e_n$. A path or circuit is simple if it does not contain the same edge more than once.

Exercise 2.41. (a) Find a simple path of length 5 and a simple path of length 6 from a to f .

(b) Find a longest path from a to f .

(c) Find a circuit of length 7.

(d) Find a circuit of length 6.



Solution: (a) ab, bc, cd, de, ef , is a path of length 5 and ab, bc, cd, de, eb, cd , is a path of length 6.

(b) one of the longest paths is the path $ad, de, ef, fb, be, ea, ab, bc, cf$ of length 9.

- (c) $ad, dc, cb, bf, fe, eb, ba$ is a circuit of length 7.
 (d) ad, dc, cf, fb, be, ea is a cycle of length 6.

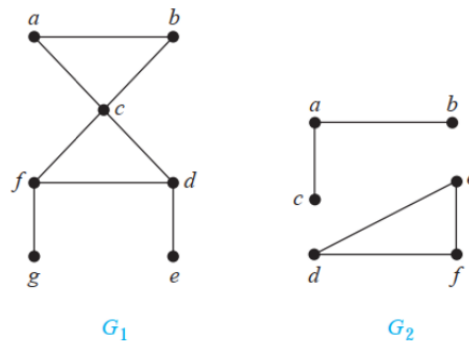
Definition 2.33. An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

A graph is connected if and only if it has exactly one component.

An undirected graph that is not connected is called disconnected.

We say that we disconnect a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

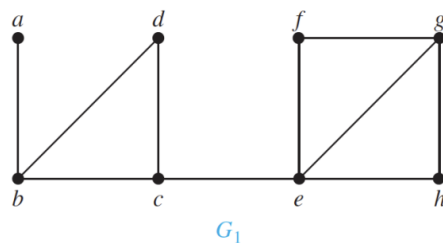
Example 2.42. The graph G_1 is connected, because for every pair of distinct vertices there is a path between them. However, the graph G_2 is not connected. For instance, there is no path in G_2 between vertices a and d .



Definition 2.34. A cut vertex is a vertex that when removed (with its boundary edges) from a graph creates more components than previously in the graph.

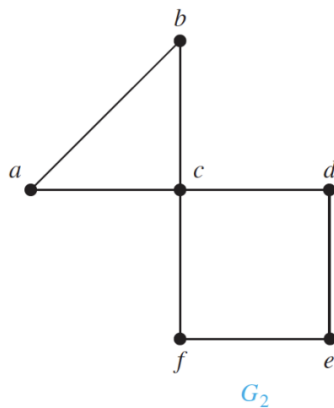
A cut edge is an edge that when removed (the vertices stay in place) from a graph creates more components than previously in the graph.

Exercise 2.43. Find the cut vertices and cut edges in the graph G_1 shown in Figure.



Solution: The cut vertices of G_1 are b, c , and e . The removal of one of these vertices (and its adjacent edges) disconnects the graph. The cut edges are $\{a, b\}$ and $\{c, e\}$. Removing either one of these edges disconnects G_1 .

Exercise 2.44. Find the cut vertices and cut edges in the graph G_2 shown in Figure.

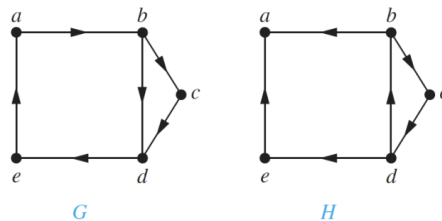


Solution: The cut vertex of G_2 is c . There are no cut edges.

Definition 2.35. A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

Exercise 2.45. *Are the directed graphs G and H in the figure strongly connected? Are they weakly connected?*

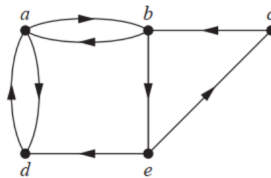


Solution: G is strongly connected because there is a path between any two vertices in this directed graph. Hence, G is weakly connected. The graph H is not strongly connected. There is no directed path from a to b . However, H is weakly connected.

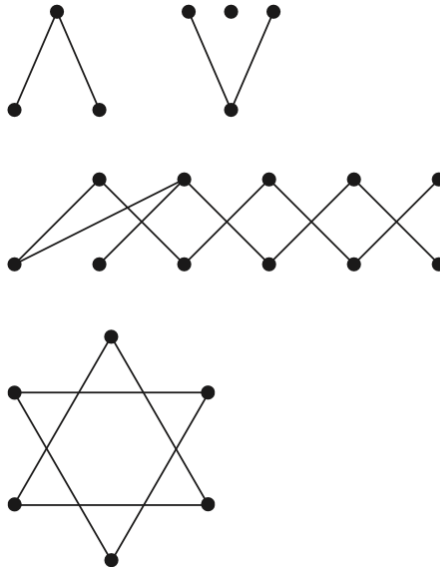
Exercises

- 1 Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

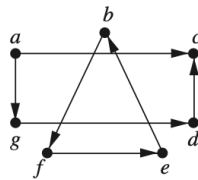
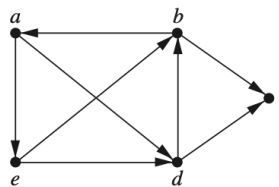
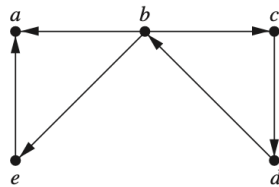
- (a) a, b, e, c, b
- (b) a, d, a, d, a
- (c) a, d, b, e, a
- (d) a, b, e, c, b, d, a



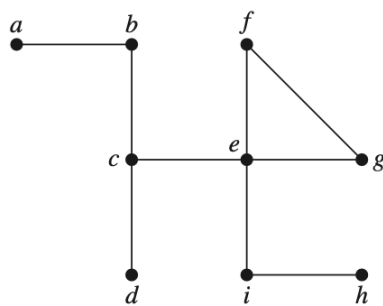
- 2 Determine whether the given graph is connected.



- 3 Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.



- 4 Find all the cut vertices of the given graph.



2.9 Euler and Hamilton Paths

Definition 2.36. (Euler Paths and Circuits)

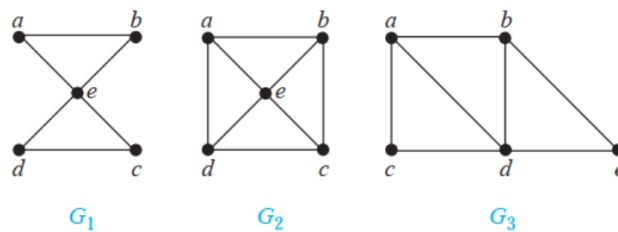
An Euler path in G is a simple path containing every edge of G .

An Euler circuit in a graph G is a simple circuit containing every edge of G .

Exercise 2.46. Determine whether the given graph has an Euler circuit.

Construct such a circuit when one exists. If no Euler circuit exists, determine

whether the graph has an Euler path and construct such a path if one exists.

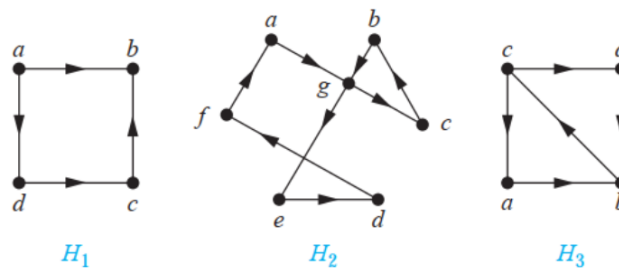


Solution: The graph G_1 has an Euler circuit, for example, a, e, c, d, e, b, a . Neither of the graph G_2 or G_3 has an Euler circuit. However, G_3 has an Euler path, namely, a, c, d, e, b, d, a, b . G_2 does not have an Euler path.

Exercise 2.47. Determine whether the given directed graph has an Euler circuit.

Construct such a circuit when one exists. If no Euler circuit exists,

determine whether the graph has an Euler path and construct such a path if one exists.

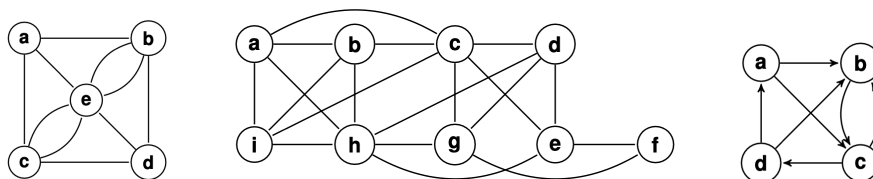


Solution: The graph H_2 has an Euler circuit, for example, $a, g, c, b, g, e, d, f, a$. Neither H_1 nor H_3 has an Euler circuit. H_3 has an Euler path, namely, c, a, b, c, d, b , but H_1 does not have an Euler path.

Theorem 2.48. *A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.*

Theorem 2.49. *A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.*

Exercise 2.50. *Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.*



Solution: By theorem 2.44, The first graph does not have an Euler circuit because we have two vertices with odd degrees (a and d). This graph does have an Euler path by Theorem 2.45. The path is as follows:

$$a, e, c, e, b, e, d, b, a, c, d$$

By theorem 2.44, the second graph has an Euler circuit because every vertex has an even degree. The circuit is as follows:

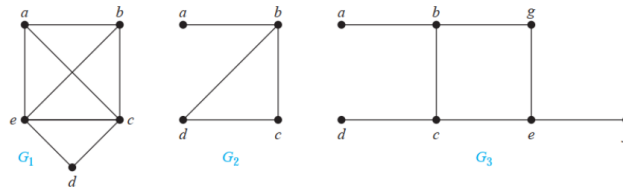
$$a, b, c, d, e, f, g, h, i, a, h, b, i, c, e, h, d, g, c, a.$$

The third graph does not have an Euler circuit because b has a different out-degree and in-degree. There is also no Euler path because b in-degree and out-degree differ by more than 1.

Definition 2.37. (Hamilton Paths and Circuits)

A simple path in a graph G that passes through every vertex exactly once is called Hamilton path and a simple circuit in a graph G that passes through every vertex exactly once is called Hamilton path.

Exercise 2.51. Which of the following simple graphs have a Hamilton circuit or, if not, a Hamilton path?



Solution: G_1 has a Hamilton circuit a, b, c, d, e, a .
 G_2 has no Hamilton circuit, but G_2 has a Hamilton path, namely, a, b, c, d .
 G_3 has neither a Hamilton circuit nor a Hamilton path.

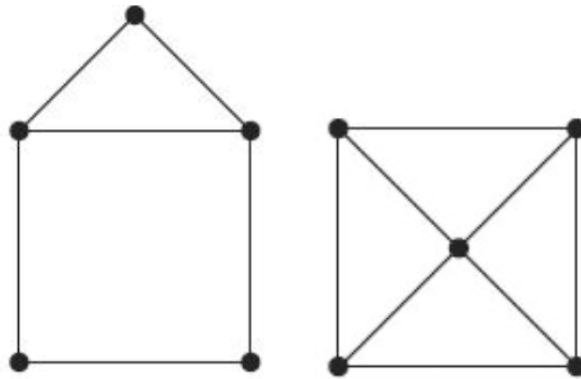
Theorem 2.52. (Dirac's Theorem)

If G is a simple graph with n vertices and $n \geq 3$ such that the degree of every vertex in G is at least $\frac{n}{2}$, then G has a Hamilton circuit.

Theorem 2.53. (Ore's Theorem)

If G is a simple graph with n vertices and $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G , then G has a Hamilton circuit.

Exercise 2.54. For each of these graphs, determine (i) whether Dirac's theorem can be used to show that the graph has a Hamilton circuit, (ii) whether Ore's theorem can be used to show that the graph has a Hamilton circuit, and (iii) whether the graph has a Hamilton circuit.

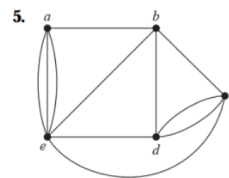
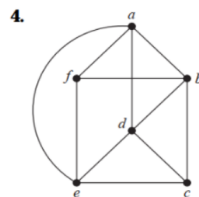
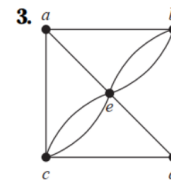
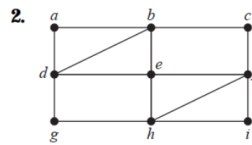
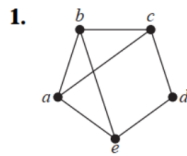


Solution: In the first graph, $n = 5$ and we cannot apply Dirac's theorem because we have vertices with degree 2 and $2 < 5/2$. We also cannot apply Ore's theorem because there are two nonadjacent vertices with degree 2 and their sum is less than n , $2 + 2 < 5$. However, we can easily see that there is a Hamilton circuit by just traversing the vertices around the pentagon. This means that both theorems do not provide necessary conditions for the existence of a Hamilton circuit.

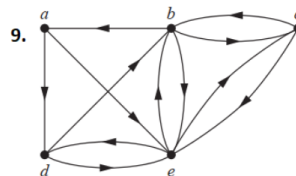
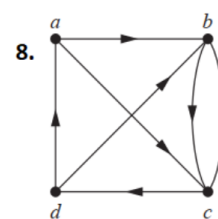
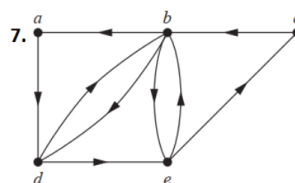
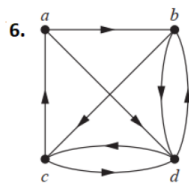
In the second graph, $n = 5$. By Dirac's theorem we have a Hamilton circuit because each vertex has a degree of 3 or 4, which is greater than $5/2$. Ore's theorem can also be applied because the smallest number for the degree of two nonadjacent vertices is 6, which is greater than or equal to 5.

Exercises

- 1 In Exercises 1-5 determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

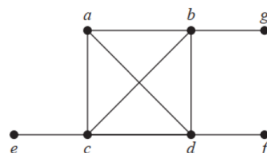
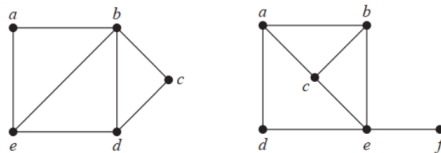


- 2 In Exercises 6-9 determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path. Construct an Euler path if one exists.



- 3 For which values of n do these graphs has an Euler circuit?
 a) K_n b) C_n c) W_n d) Q_n .

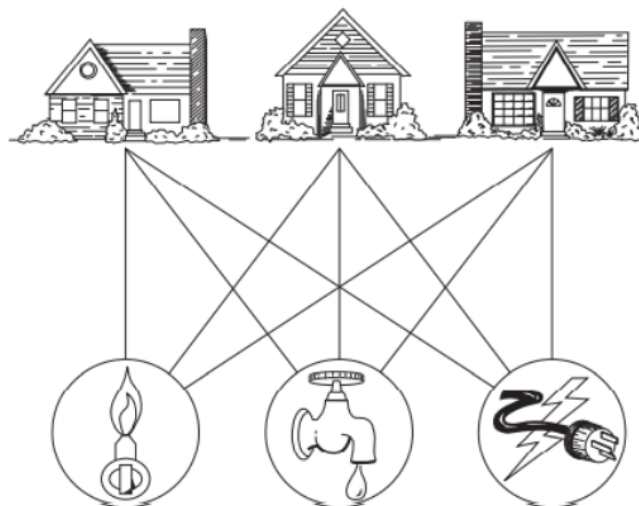
- 4 For which values of n do the graphs has an Euler path but no Euler circuit?
 a) K_n b) C_n c) W_n d) Q_n .
- 5 For which values of m and n does the complete bipartite graph $K_{m,n}$ have an
 a) Euler circuit? b) Euler path?
- 6 For the following graphs, determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.



- 7 For the above graphs, determine whether the given graph has a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

2.10 Planar Graphs

Consider the problem of joining three houses to each of the three separate utilities, as shown in the figure. Is it possible to join these three houses and utilities so that none of the connection cross?



Three Houses and Three Utilities.

This problem can be modeled using complete bipartite graph $K_{3,3}$. The original question can be rephrased as: Can $K_{3,3}$ be drawn in the plane so that no two of its edges crosses?

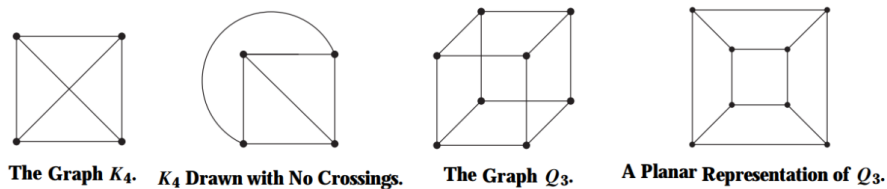
In this section we will study the question of whether a graph can be drawn in the plane without edges crossing. In particular, we will answer the houses-and-utilities problem.

Definition 2.38. (Planar graphs)

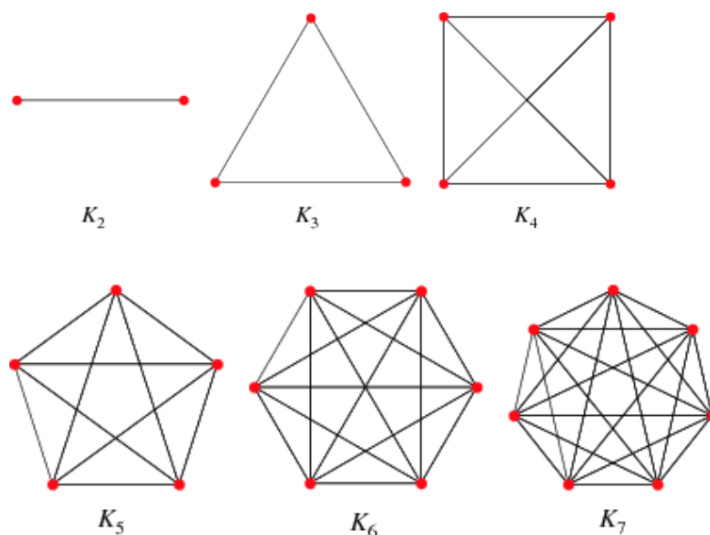
A graph is called planar if it can be drawn in the plane without any edge crossing, such a drawing is called a planar representation of the graph. A graph that is not planar is called nonplanar graph.

Exercise 2.55. *Are the graphs K_4 and Q_3 planars?*

Solution: K_4 is planar because it can be drawn without crossings. Also, Q_3 is planar because it can be drawn without any edge crossing as shown in the figure.



Exercise 2.56. For which values of n do these graph K_n is planar?



Solution: K_2 , K_3 and K_4 are planar graphs. If $n \geq 5$, then the graph K_n is non-planar.

Definition 2.39. (Regions)

A planar representation of a graph splits the plane into regions, including an unbounded region (is called exterior region).

Theorem 2.57. (Euler's Formula)

Let G be a connected planar simple graph with n vertices and q edges. Let r be the number of regions in a planar representation of G . Then, $r = q - n + 2$.

Exercise 2.58. Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

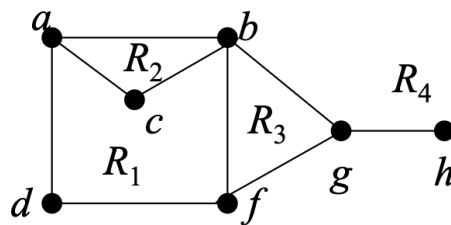
Solution: Since $\sum d(v) = 2q$, then $3(20) = 2q \implies q = 30$. Consequently, from Euler's formula, the number of regions is

$$r = q - n + 2 \implies r = 30 - 20 + 2 \implies r = 12.$$

Definition 2.40. (Degree of regions)

Let R be a region of a planar graph. We define the degree of R to be the number of edges on the boundary of R .

Exercise 2.59. Find the degree of the regions of the given graph.



Solution:

$$\deg(R_1) = 5$$

$$\deg(R_2) = 3$$

$$\deg(R_3) = 3$$

$$\deg(R_4) = 7 \quad (a - b - g - h - g - f - d - a).$$

Corollary 2.60. *If G is a connected planar simple graph with q edges and n vertices, where $n \geq 3$, then $q \leq 3n - 6$.*

Corollary 2.61. *If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.*

Proof. (By contradiction) Suppose that G is a graph such that every vertex of G has degree 6 or more. Let G have order n and size q . Then

$$2q = \sum \deg(v) \geq 6n \implies 2q \geq 6n \implies q \geq 3n > 3n - 6,$$

a contradiction, since $q \leq 3n - 6$. □

Exercise 2.62. *Show that K_5 is non-planar.*

Solution: K_5 has order $n = 5$, and size $q = 10$. Then, $q = 10 > 3(5) - 6 = 9$. But $q \leq 3n - 6$. Therefore, K_5 is non-planar.

Exercise 2.63. *Show that K_6 is non-planar.*

Solution: K_6 has order $n = 6$, and size $q = 15$. Then, $q = 15 > 3(6) - 6 = 12$. But $q \leq 3n - 6$. Therefore, K_6 is non-planar.

Corollary 2.64. *If a connected planar simple graph has q edges and n vertices with $n \geq 3$ and no circuits of length three, then $q \leq 2n - 4$.*

Exercise 2.65. *Use Corollary 2.60 to show that $K_{3,3}$ is non-planar.*

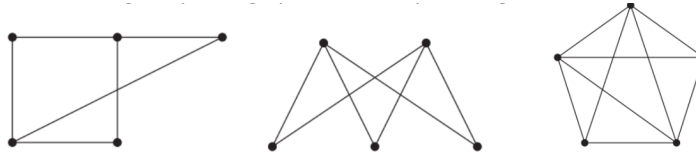
Solution: Because $K_{3,3}$ has no circuits of length three (this is easy to see because it is bipartite), Corollary 2.60 can be used. $K_{3,3}$ has 6 vertices and 9 edges. Because $q = 9 > 2(6) - 4 = 8$, Corollary 2.60 shows that $K_{3,3}$ is non-planar.

Exercise 2.66. *Use Corollary 2.60 to show that $K_{3,4}$ is non-planar.*

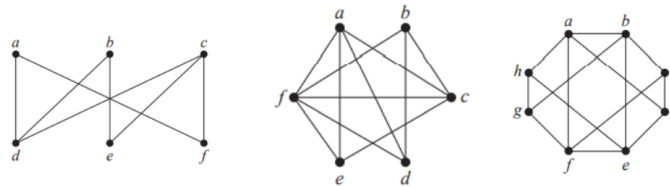
Solution: Because $K_{3,4}$ has no circuits of length three (this is easy to see because it is bipartite), Corollary 2.60 can be used. $K_{3,4}$ has 7 vertices and 12 edges. Because $q = 12 > 2(7) - 4 = 10$, Corollary 2.60 shows that $K_{3,4}$ is non-planar.

Exercises

- 1 Can five houses be connected to two utilities without connections crossing?
- 2 Draw the given planar graph without any crossings.



- 3 Determine whether the given graph is planar. If so, draw it so that no edges cross.



- 4 Suppose that a connected planar graph has 8 vertices, each of degree 3. Into how many regions is the plane divided by a planar representation of this graph?
- 5 Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?
- 6 The vertices of a certain graph G have degrees 3, 4, 4, 4, 5, 6, 6. Prove that G is non-planar.
- 7 Which of these nonplanar graphs have the property that the removal of any vertex and all edges incident with that vertex produces a planar graph?
 - a) K_5
 - b) K_6
 - c) $K_{3,3}$
 - d) $K_{3,4}$

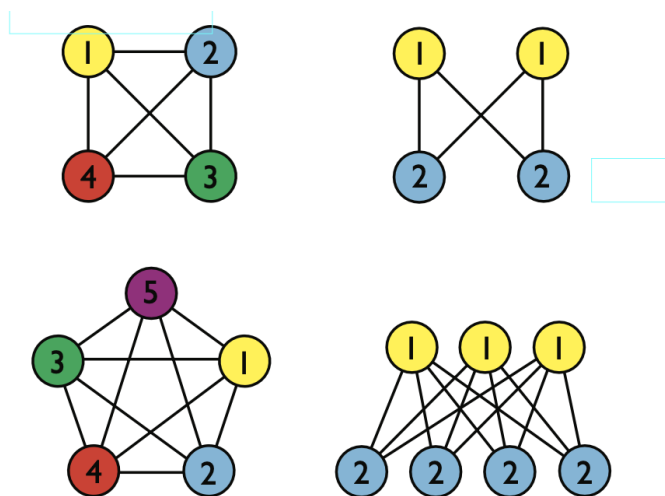
2.11 Graph Coloring

Definition 2.41. (Graph Coloring)

A k -coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color. For examples of graphs and colourings include

- 1 K_n , the complete graph on n vertices is clearly n -colourable, but not $(n - 1)$ colourable.
- 2 $K_{m,n}$, the complete bipartite graph on groups of m and n vertices, is 2-colourable.

Both classes of example are illustrated in the below Figure:



The complete graphs K_4 and K_5 as well as the complete bipartite graphs $K_{2,2}$ and $K_{3,4}$, each coloured using the smallest possible number of colours.

Definition 2.42. The chromatic number of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph G is denoted by $\chi(G)$. (Here χ is the Greek letter chi). Thus, as the examples above suggest, $\chi(K_n) = n$ and $\chi(K_{m,n}) = 2$.

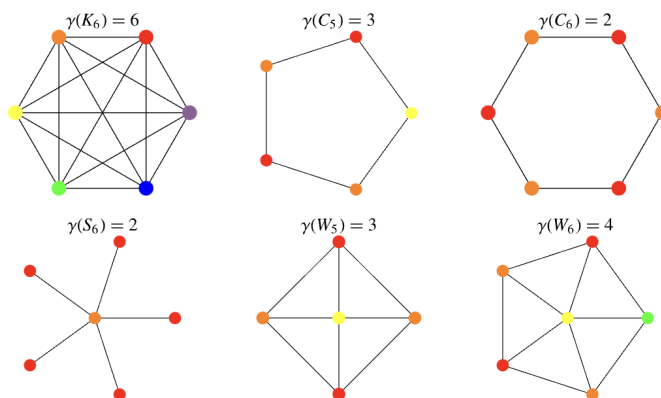
Theorem 2.67. *The chromatic number of a planar graph is no greater than four.*

Corollary 2.68. *If H is a subgraph of G and G is k -colourable, then so is H .*

Corollary 2.69. *If H is a subgraph of G then $\chi(H) \leq \chi(G)$.*

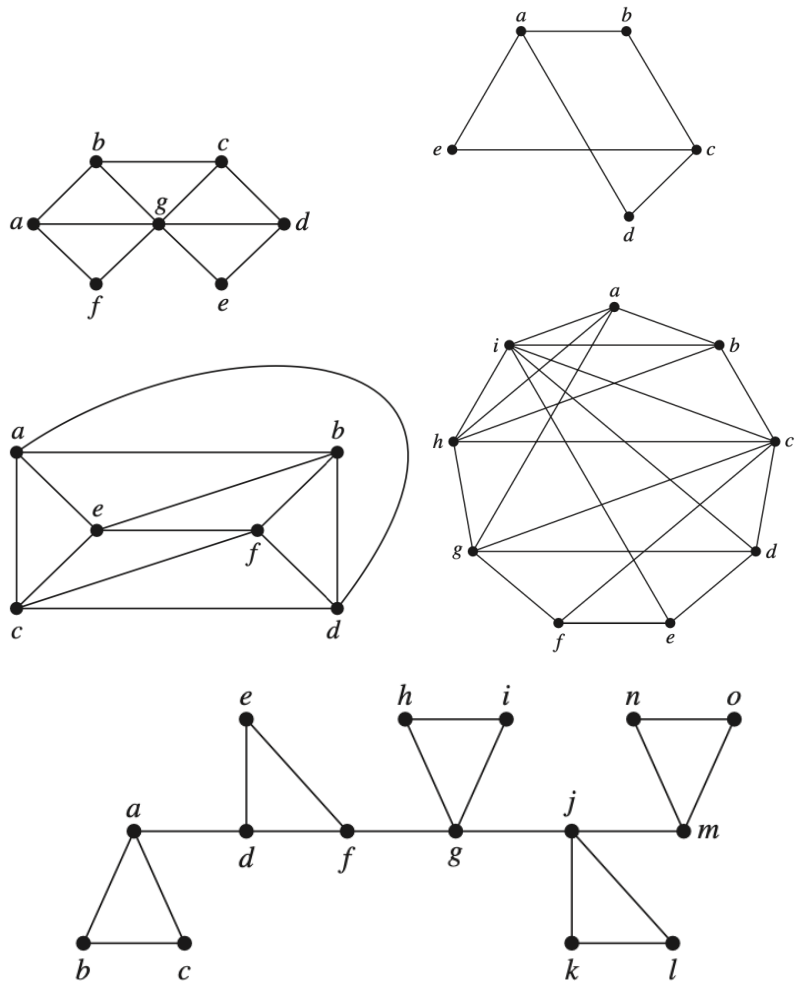
The following table gives the chromatic numbers for some named classes of graphs.

graph G	$\chi(G)$
complete graph K_n	n
cycle graph $C_n, n > 1$	$\begin{cases} 3 & \text{for } n \text{ odd} \\ 2 & \text{for } n \text{ even} \end{cases}$
star graph $S_n, n > 1$	2
wheel graph $W_n, n > 2$	$\begin{cases} 3 & \text{for } n \text{ odd} \\ 4 & \text{for } n \text{ even} \end{cases}$



Exercises

- Find the chromatic number of the given graph.

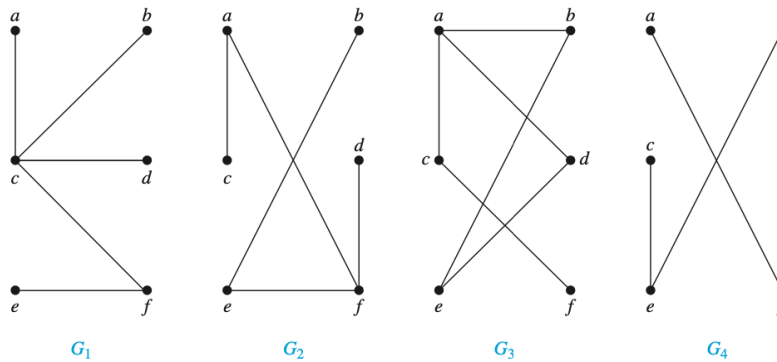


3 TREES

Definition 3.1. (Tree)

A tree is a connected undirected graph with no simple circuits.

Exercise 3.1. Which of the graphs shown below are trees?

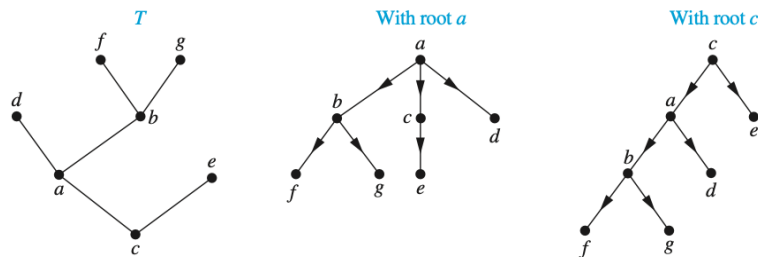


Solution: G_1 and G_2 are trees, because both are connected graphs with no simple circuits. G_3 is not a tree because e, b, a, d, e is a simple circuit in this graph. Finally, G_4 is not a tree because it is not connected.

Theorem 3.2. An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Definition 3.2. (Rooted tree)

A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

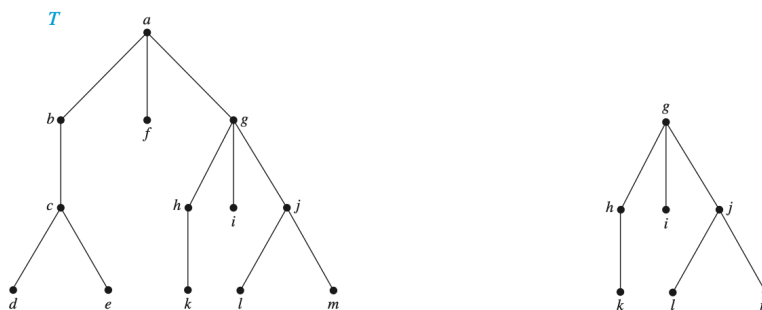


Terminology for trees

Suppose that T is a rooted tree.

- If v is a vertex in T other than the root, the parent of v is the unique vertex u such that there is a directed edge from u to v .
- When u is the parent of v , v is called a child of u .
- Vertices with the same parent are called siblings.
- The ancestors of a vertex other than the root are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root.
- The descendants of a vertex v are those vertices that have v as an ancestor.
- A vertex of a rooted tree is called a leaf if it has no children.
- Vertices that have children are called internal vertices.
- If a is a vertex in a tree, the subtree with a as its root is the subgraph of the tree consisting of a and its descendants and all edges incident to these descendants.

Example 3.3.



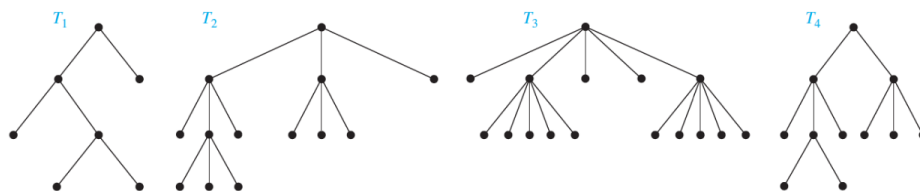
- The parent of c is b .
- The children of g are h , i , and j .

- The siblings of h are i and j .
- The ancestors of e are c, b , and a .
- The descendants of b are c, d , and e .
- The internal vertices are a, b, c, g, h , and j .
- The leaves are d, e, f, i, k, l , and m .
- The subtree rooted at g is shown above.

Definition 3.3. (m -ary tree)

A rooted tree is called an m -ary tree if every internal vertex has no more than m children. The tree is called a full m -ary tree if every internal vertex has exactly m children. An m -ary tree with $m = 2$ is called a binary tree.

Exercise 3.4. Are the given trees, full m -ary tree for some positive integer m ?



Solution:

- T_1 is a full binary tree because each of its internal vertices has two children.
- T_2 is a full 3-ary tree because each of its internal vertices has three children.
- T_3 each internal vertex has five children, so T_3 is a full 5-ary tree.
- T_4 is not a full m -ary tree for any m because some of its internal vertices have two children and others have three children.

Properties of Trees

Theorem 3.5. *A tree with n vertices has $n - 1$ edges.*

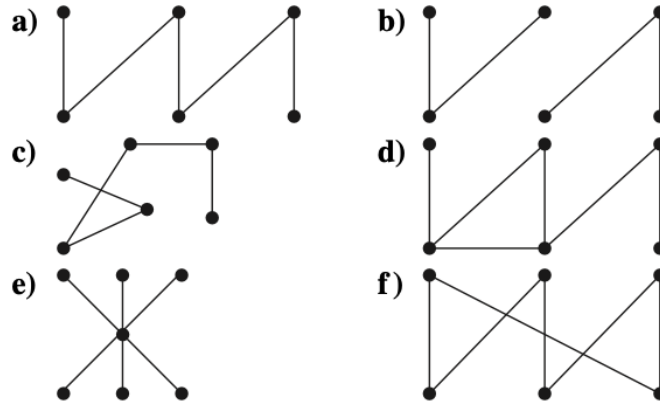
Theorem 3.6. *A full m -ary tree with i internal vertices contains $n = mi + 1$ vertices.*

Theorem 3.7. *A full m -ary tree with*

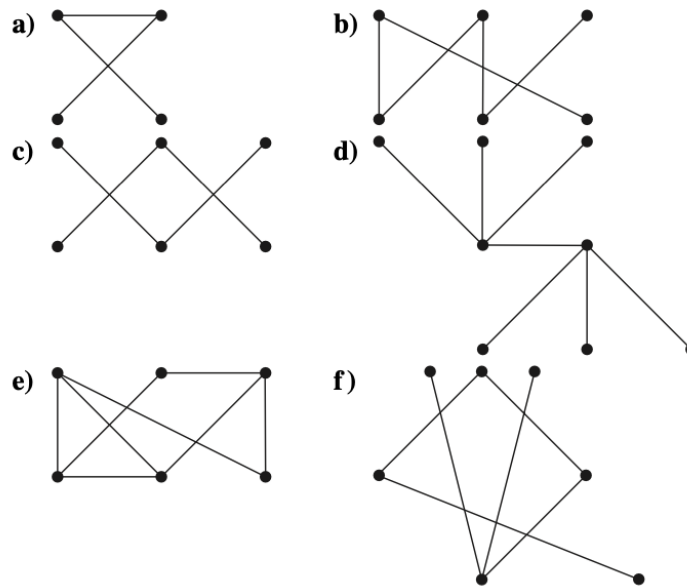
- *n vertices has $i = (n - 1)/m$ internal vertices and $l = [(m - 1)n + 1]/m$ leaves,*
- *i internal vertices has $n = mi + 1$ vertices and $l = (m - 1)i + 1$ leaves,*
- *l leaves has $n = (ml - 1)/(m - 1)$ vertices and $i = (l - 1)/(m - 1)$ internal vertices.*

Exercises

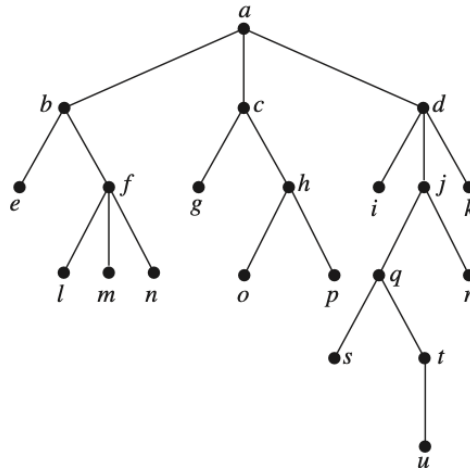
1 Which of these graphs are trees?



2 Which of these graphs are trees?



3 Answer these questions about the rooted tree illustrated.

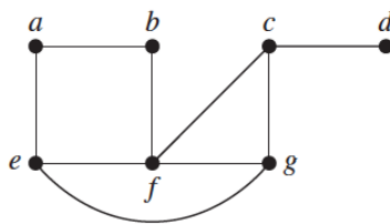


- a) Which vertex is the root?
 - b) Which vertices are internal?
 - c) Which vertices are leaves?
 - d) Which vertices are children of j ?
 - e) Which vertex is the parent of h ?
 - f) Which vertices are siblings of o ?
 - g) Which vertices are ancestors of m ?
 - h) Which vertices are descendants of b ?
- 4 Is the rooted tree in Exercise 3 a full m -ary tree for some positive integer m ?
 - 5 Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?
 - 6 How many edges does a tree with 10,000 vertices have?
 - 7 How many vertices does a full 5-ary tree with 100 internal vertices have?
 - 8 How many edges does a full binary tree with 1000 internal vertices have?
 - 9 How many leaves does a full 3-ary tree with 100 vertices have?

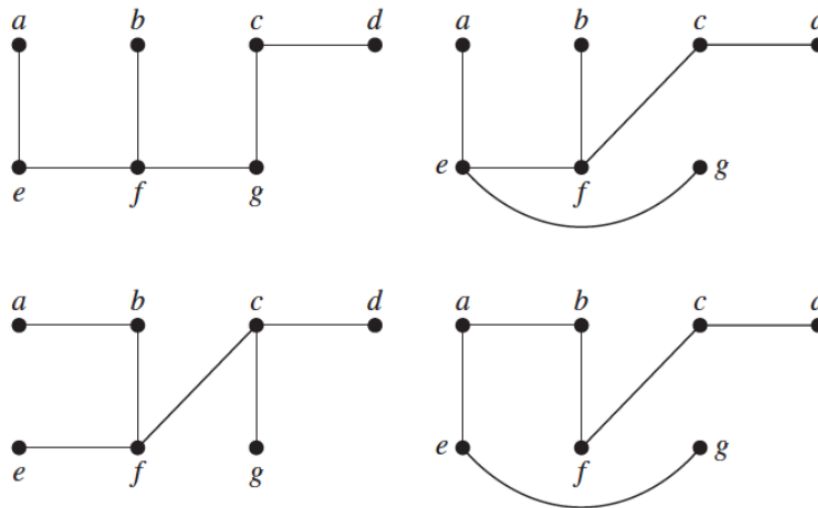
3.1 Spanning tree

Definition 3.4. Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G .

Exercise 3.8. Find a spanning tree of the simple graph G shown in Figure.



Solution:

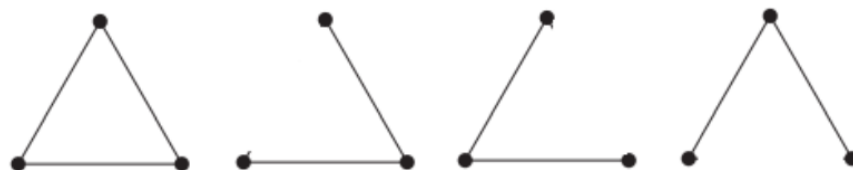


Theorem 3.9. A simple graph is connected if and only if it has a spanning tree.

The number of Spanning trees

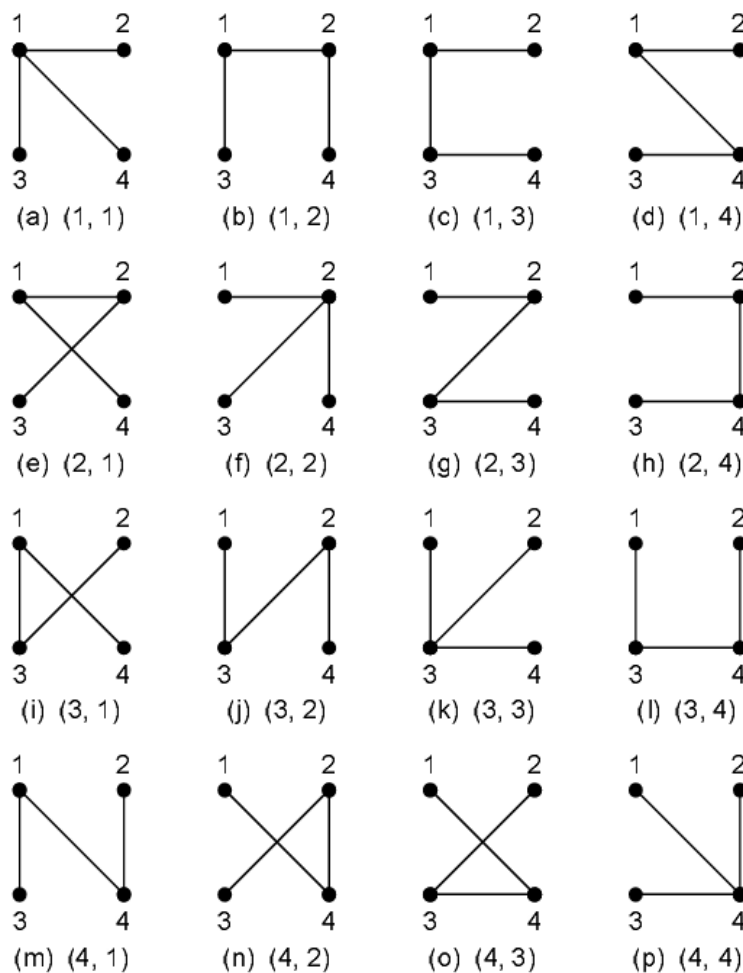
Exercise 3.10. Find all the spanning trees of the graph K_3 .

Solution: K_3 containing 3 spanning trees.



Exercise 3.11. Find all the spanning trees of K_4 .

Solution: K_4 containing 16 spanning trees.



Theorem 3.12. *The number of spanning trees of K_n is n^{n-2} .*

3.2 Cayley Tree Formula

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} \end{bmatrix}$$

By a cofactor of an $n \times n$ matrix $A = [a_{ij}]$, we mean $(-1)^{i+j} \det(A_{ij})$, where $\det(A_{ij})$ indicates the determinant of the $(n-1) \times (n-1)$ submatrix A_{ij} which obtaining by deleting row i and column j of A .

Example 3.13. $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 2 & 3 & -2 \\ 4 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}.$

The $(1,1)$ cofactor of A is

$$\begin{aligned} (-1)^{1+1} \det \begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} &= 2 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \\ &= 2(3 - 1) - 3(6 - 0) + (-2)(2 - 0) = -18 \end{aligned}$$

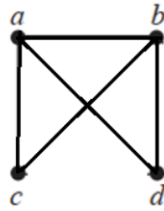
The $(2, 1)$ cofactor of A is

$$\begin{aligned}
 (-1)^{2+1} \det \begin{bmatrix} 2 & -1 & 4 \\ 2 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} &= - \left(2 \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} - (-1) \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + 4 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right) \\
 &= -(2(3-1) - (-1)(6-0) + 4(2-0)) = -18
 \end{aligned}$$

Theorem 3.14. (Kirchhoff Theorem (Matrix Tree Theorem))

Let M be the matrix obtained from the adjacency matrix of a simple connected graph G by changing all '1' to -1' and each diagonal 0 to the degree of the corresponding vertex. Then the number of spanning trees of G is the number of any cofactor of M .

Exercise 3.15. Find the number of spanning trees of the given graph.

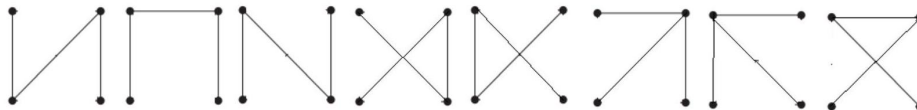


Solution: The adjacency matrix is:

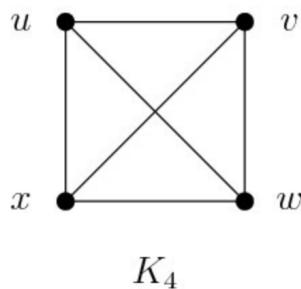
$$M = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

The $(1,1)$ cofactor of M is $\det \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = 8$

The $(2, 1)$ cofactor of $M = (-1)^{2+1} \det \begin{bmatrix} -1 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = 8$



Exercise 3.16. Use Krichooff theorem to find number the spanning trees of K_4 .



Solution: The adjacency matrix of K_4 .

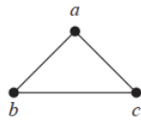
$$A(K_4) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$\begin{aligned} \text{The } (1, 1) \text{ cofactor of } M \text{ is } \det \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \\ = 3 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ -1 & 3 \end{vmatrix} + (-1) \begin{vmatrix} -1 & 3 \\ -1 & -1 \end{vmatrix} \\ = 3(9 - 1) + (-3 + 1) - (1 + 3) = 24 - 2 - 4 = 16. \end{aligned}$$

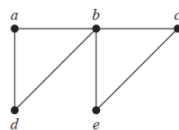
Exercises

- 1 In Exercises 1-5 find a spanning tree for the graph shown by removing edges in simple circuits.

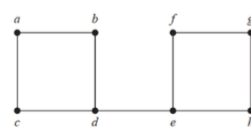
1.



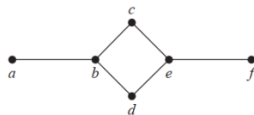
2.



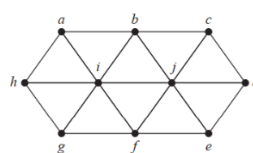
3.



4.



5.



- 2 Use Krichooff theorem to find number the spanning trees of:

- a) K_5
- b) $K_{4,4}$
- c) $K_{1,6}$
- d) Q_3
- e) C_5
- f) W_5