

Analytical Geometry (Math 241)

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Chapter (1): The general equation of first degree in x, y

A. Cartesian coordinates:

By Cartesian coordinates, we mean the general two axes $X - axis, Y - axis$ in the $XY - plane$.

Any point in the $XY - plane$ is defined during its value x through the $X - axis$ and its value y through the $Y - axis$, denoted by the pair (x, y) .

Distance between two points:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two given points, the distance between them in terms of their coordinates is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example (1.1): Find the distance between the two points $(1, -4)$ and $(0, -1)$.

Solution:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \rightarrow$$

$$d = \sqrt{(0 - 1)^2 + ((-1) - (-4))^2} \rightarrow$$

$$d = \sqrt{10}.$$

Example (1.2): Find the distance between the two points $(6, 3)$ and $(12, -2)$.

Solution:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \rightarrow$$

$$d = \sqrt{(12 - 6)^2 + ((-2) - (3))^2} \rightarrow$$

$$d = \sqrt{61}.$$

The ratio formula:

To find the Cartesian coordinates of a point that divides the segment joining two given points with a given ratio:

let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any given points, and let the point $R(x, y)$ divides internally the segment joining P, Q with the ratio $m_1 : m_2$. Then, The coordinates of the point $R(x, y)$ are:

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \quad y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}.$$

Note that: The coordinates of the point $R(x, y)$ in the middle of a line joining the points (x_1, y_1) and (x_2, y_2) are:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

If the point $R(x, y)$ divides externally the segment PQ with the ratio $m_1 : m_2$, then the coordinates of the point $R(x, y)$ are:

$$\left(x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \quad y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2} \right)$$

Example (1.3): Find the coordinates of the point which divides internally the line segment joining the points $(4, -3)$ and $(8, 5)$ in the ratio $3 : 1$.

Solution:

$$(x, y) = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right) \rightarrow$$

$$(x, y) = \left(\frac{24+4}{4}, \frac{15-3}{4} \right) = \left(\frac{28}{4}, \frac{12}{4} \right) = (7, 3).$$

Example (1.4): Find the coordinates of the point which divides externally the line segment joining the points $(3, 5)$ and $(6, 7)$ in the ratio $3 : 2$.

Solution:

$$(x, y) = \left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2} \right) \rightarrow$$

$$(x, y) = \left(\frac{18-6}{1}, \frac{21-10}{1} \right) = \left(\frac{12}{1}, \frac{11}{1} \right) = (12, 11).$$

(*) The area of a triangle:

During some a three points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) that represents three vertices of a triangle, then the area of this triangle is computed as follows:

$$Area = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Note that: If the value of the area was negative, it only means that the order of the vertices was not correct, and so the area is the absolute value of the above determinant divided by 2.

Three points are called collinear if these points do not form a real triangle, that is, the area is in before is equal to zero. Thus, the condition of being three points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) collinear is:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Example (1.5): Find the area of the triangle whose vertices are $(2, 1)$, $(5, -3)$, $(-8, 0)$.

Solution:

$$Area = \frac{1}{2} \begin{vmatrix} 2 & 1 & 1 \\ 5 & -3 & 1 \\ -8 & 0 & 1 \end{vmatrix} = \frac{1}{2} (-6 - 8 - 5 - 24) = -\frac{43}{2}.$$

Thus, the area is $\frac{43}{2}$.

Example (1.6): Show that the three points $A(1, 5)$, $B(6, -1)$, $C(-4, 11)$ lie on a line (collinear).

Solution:

$$Area = \frac{1}{2} \begin{vmatrix} 1 & 5 & 1 \\ 6 & -1 & 1 \\ -4 & 11 & 1 \end{vmatrix} = \frac{1}{2} (0) = 0.$$

Since the area is zero, then the three points did not form any triangle, that is, they are on a line (collinear).

Example (1.7): Find the value of λ so that the points $(\lambda, 2)$, $(-2, 1)$, $(-3, -2)$ are collinear.

Solution:

Since

$$Area = \frac{1}{2} \begin{vmatrix} \lambda & 2 & 1 \\ -2 & 1 & 1 \\ -3 & -2 & 1 \end{vmatrix} = \frac{1}{2} [\lambda(3) + 2(-1) + 1(7)] = \frac{1}{2} [3\lambda + 5] = 0 \rightarrow$$
$$\lambda = -\frac{5}{3}.$$

B. The locus of a curve:

(*) Equation and locus:

The two fundamental problems of analytical geometry are:

- 1- Given an equation, how to sketch or describe the corresponding locus.**
- 2- Given a locus defined by some geometrical conditions, how to find the corresponding equation.**

The three important things to draw a geometrical figure (locus or graph) are:

- 1- Intercepts.**
- 2- Symmetry.**
- 3- Extent.**

Intercepts:

To determine the X-intercept, let $y = 0$ in the equation and solve for x . Similarly, to find the Y-intercept, let $x = 0$ in the equation and solve for y .

Thus, for example:

For the equation $y^2 + 2x = 16$, if $y = 0$ then $x = 8$, and if $x = 0$ Then $y = \pm 4$.

Hence, the X -intercept of the $X - axis$ of that curve is 8 and the Y -intercept over the $Y - axis$ are ± 4 .

Symmetry:

1- If the equation remains unchanged when x is replaced with $(-x)$, the graph is symmetric with respect to the $Y - axis$.

As an example: Let $x^2 - 6y + 12 = 0$.

In this equation if x is replaced with $(-x)$, the equation remains unchanged. There are two values of x for each value of y as follows:

$$x = \pm \sqrt{6y - 12}.$$

2- If the equation remains unchanged when y is replaced with $(-y)$, the graph is symmetric with respect to the $X - axis$.

As an example: Let $y^2 - 4x - 7 = 0$.

In this equation if y is replaced with $(-y)$, the equation remains unchanged. There are two values of y for each value of x as follows:

$$y = \pm \sqrt{4x + 7}.$$

3- If an equation remains unchanged when x is replaced with $(-x)$, and y is replaced with $(-y)$, then the graph is symmetric with respect to the origin. For example: Let $x^3 + x + y^3 = 0$.

Extent: By the extent, we mean the domain of the graph or the locus.

Consider the equation $y^2 = 2x - 3$ or $y = \pm\sqrt{2x - 3}$. Here, x must satisfy that $x \geq \frac{3}{2}$ to have a real image or a real graph corresponding to each value of x . So, the extent of the given equation is the interval $[\frac{3}{2}, \infty)$.

Now, the locus of a given equation:

Example (1.8): Discuss and plot the locus of the ellipse $9x^2 + 16y^2 = 144$.

Solution:

Intercepts:

When $y = 0$, then $x = \pm 4$, and if $x = 0$ then $y = \pm 3$. Hence, the X -intercepts are ± 4 and the Y -intercepts are ± 3 .

Symmetry:

Since the equation contains only even powers of x and y , then the curve is symmetric about both axes and therefore about the origin. Hence, it is sufficient to plot only that part of the curve in the first quadrant and determine the rest of the curve by symmetry.

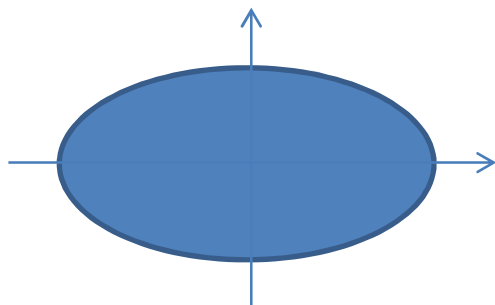
Extent:

Solving for y and for x :

$$y = \pm \frac{3}{4} \sqrt{16 - x^2}, \quad x = \pm \frac{4}{3} \sqrt{9 - y^2}.$$

Hence, the extent of the graph is the values of x as: $-4 \leq x \leq 4$ and then $-3 \leq y \leq 3$.

x	0	± 1	± 2	± 3	± 4
y	± 3	± 2.9	± 2.6	± 2	0



Homework (1.1):

- (a) Find the value of α so that the points $(1, 2)$, $(-1, \alpha)$, $(3, -3)$ are collinear.
- (b) Discuss and plot the locus of the ellipse $4x^2 + y^2 = 9$.
- (c) Find the coordinates of the point which divides externally the line segment joining the points $(4, 3)$ and $(2, 1)$ in the ratio $1 : 2$.
- (d) Find the coordinates of the point which divides internally the line segment joining the points $(3, -1)$ and $(-2, 2)$ in the ratio $2 : 2$.

C. The straight line:**(*)Particular Lines:**

- The equation of the line parallel to $Y - axis$ which is at distance $= a$.
- The equation of the line parallel to $X - axis$ which is at distance $= b$.

Corollary:

The equation of the $X - axis$ is $y = 0$ and that of $Y - axis$ is $x = 0$.

(*)Slope of a line:

The slope of a line joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$m = \tan \alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

where α is the angle that the line PQ makes with the positive direction of $X - axis$.

(*) Parallel and perpendicular lines:

Two lines are said to be parallel if they make the same angle with $-axis$, and hence they have the same slope. Conversely, if two lines has the same slope, then they make the same angle with $X - axis$ and hence they are parallel.

Thus, if m and m' be the slopes of any two parallel lines, then

$$m = m' .$$

Note: The lines parallel to $X - axis$ have slope zero while the lines parallel to $Y - axis$ have the slope ∞ . Further, consider any two perpendicular lines with slopes m, m' and making angles α, β with $X - axis$ respectively. Then

$$\tan \alpha = m , \quad \tan \beta = m' ,$$

$$\tan \alpha \cdot \tan \beta = -1 \rightarrow$$

$$m \cdot m' = -1 .$$

Conversely, if $m \cdot m' = -1$, then the lines with slopes m, m' are perpendicular to each other.

(*) Various forms of the equation of a line:

There are five standard forms of the equation of lines as follows.

(a) Slope-Intercept form:

Let AB be a line having slope m and making k as $Y - intercept$ on $Y - axis$.

Let α be the angle that the line AB makes with the positive direction of $X - axis$ so that $m = \tan \alpha$, then the equation of the line AB is:

$$y = mx + k .$$

Note: The equation of the line passing through the origin is $y = mx$.

(b) Intercept form:

Let AB be the line which have $X - intercepts$, $Y - intercepts$ as a and b , on $X - axis$ and $Y - axis$ respectively.

Let $p(x, y)$ be any point on . Then the equation of the line AB is

$$\frac{x}{a} + \frac{y}{b} = 1 .$$

(c) Normal form:

The equation of a line in terms of the length of the perpendicular p on it from the origin and the angle α by which that the perpendicular makes with $X - axis$ is

$$x \cos \alpha + y \sin \alpha = p .$$

This is the equation of the line AB .

Note: Normal form is also referred to "perpendicular form".

(d) Point-Slope form:

To find the equation of a line which passes through a give point $A(x_1, y_1)$ and makes an angle with the $X - axis$, let m be the slope of the line passing through $A(x_1, y_1)$ and let $p(x, y)$ be any point on the line. Then

$$m = \frac{y-y_1}{x-x_1} .$$

Hence, the equation of the line passing through the point (x_1, y_1) and having slope m is

$$y - y_1 = m (x - x_1) .$$

(e) Two-point form:

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given points. Since the points A and B both lie on the line, then the slope m of the line is given by

$$m = \frac{y_2-y_1}{x_2-x_1} ,$$

and again the equation of the line through the point $A(x_1, y_1)$ having the previous slope m is

$$y - y_1 = m (x - x_1) .$$

Hence, the equation of the line through the points A and B is

$$y - y_1 = \left(\frac{y_2-y_1}{x_2-x_1} \right) (x - x_1) .$$

(*) The perpendicular distance from a point to a line:

For a line $ax + by + c = 0$ and a point (x_1, y_1) , we have the distance from a point (x_1, y_1) to that line is computed as the absolute value of the value

$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$, and then it is given by:

$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|.$$

(*) Reduction of the general equation of a line to standard forms:

The general of equation of first degree in x and y of a line is

$$ax + by + c = 0 \dots\dots\dots (1)$$

where a, b, c are constants.

This equation represents a line that all various standard forms could be obtained. Here we will produce three standard forms from the general form as follows.

(i) Slope -Intercept form:

Equation (1) could be written as

$$y = -\frac{a}{b}x - \frac{c}{b}, \quad b \neq 0$$

which is of the form $y = mx + k$ where $m = -\frac{a}{b}$ and $k = -\frac{c}{b}$.

This is the slope –intercept form of the line

Note: The slope of the line $ax + by + c = 0$ is given by

$$m = -\frac{a}{b} = -\frac{\text{coeff. of } x}{\text{coeff. of } y}.$$

(ii) Intercept form:

On dividing equation (1) by $(-c)$, $c \neq 0$, we get

$$\frac{a}{-c}x + \frac{b}{-c}y - 1 = 0 \rightarrow$$

$$\frac{x}{-c/a} + \frac{y}{-c/b} = 1$$

which is the intercept form of the line, clearly $\frac{-c}{a}$ and $\frac{-c}{b}$ are the intercepts on both of X – axis and Y – axis .

(iii) Normal form:

Equation (1) could be written as

$$\frac{a}{\sqrt{a^2+b^2}} x + \frac{b}{\sqrt{a^2+b^2}} y = -\frac{c}{\sqrt{a^2+b^2}}.$$

This equation is of the form

$$x \cos \alpha + y \sin \alpha = p$$

which is the normal form, where

$$\cos \alpha = \frac{a}{\sqrt{a^2+b^2}} \quad , \quad \sin \alpha = \frac{b}{\sqrt{a^2+b^2}} \quad , \quad p = \frac{c}{\sqrt{a^2+b^2}} .$$

(*) Angle between two lines:

Let the equations of the lines are $y = m x + k$ and $y = m' x + k$ making angles θ and θ' , respectively with $X - axis$, therefore $m = \tan \theta$ and $m' = \tan \theta'$,

$$\tan(\theta - \theta') = \frac{\tan \theta - \tan \theta'}{1 + \tan \theta \tan \theta'} = \frac{m - m'}{1 + m m'} .$$

Then the required angle is:

$$\varphi = \tan^{-1} \left(\frac{m - m'}{1 + m m'} \right) .$$

(*) Bisectors of angles:

The bisectors of the angle between two lines are the locus of a point which moves so that the perpendicular distances of the point from the lines are equal in magnitude.

$$\text{Let } a_1 x + b_1 y + c_1 = 0 \quad \text{and} \quad a_2 x + b_2 y + c_2 = 0$$

be any two lines intersecting in $\mathfrak{R}$.

The equations of the lines which bisect the angle between the two given lines are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} .$$

which are the equation of bisector (with positive sign) called internal bisector and the equation of bisector (with negative sign) called external bisector where they are perpendicular to each other.

(*) Concurrent lines:

Let the three lines are

$$\begin{aligned} a_1 x + b_1 y + c_1 &= 0 , \\ a_2 x + b_2 y + c_2 &= 0 , \\ a_3 x + b_3 y + c_3 &= 0 . \end{aligned}$$

Then, the three lines are called concurrent lines if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 .$$

Homework (1.2):

- 1) Find the equation of the line joining the points $(3, 2)$, $(-1, -5)$, and find the x – *intercepts* and y – *intercepts* .
- 2) Find the equation of the line through $(2, -1)$ and perpendicular to $4x - 3y = 5$.
- 3) Reduce the general equation $7x - 2y = 8$ into the standard forms.
- 4) Find the equation of the line through $(-2, 3)$ which makes an angel 45° with the line $3x + y + 5 = 0$.
- 5) Find the equations of bisectors of the angel between the two lines $12x + 5y - 4 = 0$, $3x + 4y + 7 = 0$.
- 6) Find the equation of the line which is perpendicular to $3x + 8y - 5 = 0$ and which passes through the point $(2, -1)$.
- 7) Show that the lines: $4x + 7y - 9 = 0$, $5x - 8y + 15 = 0$, $9x - y + 6 = 0$ are concurrent.
- 8) Find the equation of the line passing through the point $(2, 4)$ and the point of intersection of the lines $2x - y = 4$, $3x + 2y = 13$.
- 9) Find the equation of the line which passes through the point $(3, -1)$ and makes equal intercepts on the X-axis and Y-axis.
- 10) Reduce the general equation $x + y + 4 = 0$ into the standard forms.
- 11) Reduce the general equation $3x + 4y = 1$ into the standard forms.
- 12) Reduce the general equation $2x + 5y + 13 = 0$ into the standard forms.
- 13) Calculate the perpendicular distance of the point $(-3, -4)$ from the line $12x - 5y + 82 = 0$.
- 14) Calculate the perpendicular distance of the point $(1, -2)$ from the line $3x - 4y = 2$.

15) Prove that the three lines $5x + 3y - 7 = 0$, $3x - 4y - 10 = 0$, $x + 2y = 0$ are concurrent.

D. Polar coordinates and transformations:

The Cartesian coordinates and the polar coordinates have transformation equations to change from one to another.

$$x = r \cos \theta \quad , \quad y = r \sin \theta \quad , \quad r^2 = x^2 + y^2 \quad , \quad \theta = \cos^{-1} \left(\frac{x}{r} \right) .$$

Example (1.9): Transfer the following points into polar coordinates:

$(1, 3)$, $(2, 5)$, $(-1, -3)$.

Solution: In class.

Example (1.10): Transfer the following points into Cartesian coordinates:

$(1, 30^\circ)$, $(2, 50^\circ)$, $(1, -30^\circ)$, $(2, -180^\circ)$, $(1, 90^\circ)$.

Solution: In class.

Example (1.11): Transfer the following equations into Polar coordinates:

$$4x + y^2 = 2 \quad , \quad 2x^2 + 2y^2 + 4x + 4y = 6 .$$

Solution: In class.

Example (1.12): Transfer the following equations into Cartesian coordinates:

$$r^2 = 1 + \cos \theta , \quad r^2 - r \sin \theta = \tan \theta , \quad 4 \sec^2 \theta + 3 = r^2 .$$

Solution: In class.

Chapter (2): The general equation of second degree in x, y

A. A pair of lines:

A General equation of second degree in x, y is given by

$$a x^2 + 2h x y + b y^2 + 2g x + 2f y + c = 0 ,$$

where $\{a, b, c, h, f, g\}$ are constants.

If $\{f, g, c\}$ are all zeros, then the equation will be homogenous equation of degree 2, and will be written in the form:

$$a x^2 + 2h x y + b y^2 = 0 .$$

In this homogenous case, The two equations of the two lines of the pair of lines are given as follows:

$$y = m_1 x \quad \& \quad y = m_2 x ,$$

where

$$m_1 = \frac{-h + \sqrt{h^2 - ab}}{b} \quad \& \quad m_2 = \frac{-h - \sqrt{h^2 - ab}}{b} .$$

We can write the pair of lines as follow:

$$y = \left(\frac{-h + \sqrt{h^2 - ab}}{b} \right) x \quad \& \quad y = \left(\frac{-h - \sqrt{h^2 - ab}}{b} \right) x$$

Note that: If $h^2 - a b < 0$, then the pair is imaginary pair of lines,

If $h^2 - a b = 0$, then the pair consists of real lines but parallel or identical lines, they have the same slope,

If $h^2 - a b > 0$, then the pair consists of real lines and they are distinct, they have different slopes.

Example (2.1): Let $6x^2 - 5 x y + y^2 = 0$. Find the equation of the pair of lines.

Solution:

By comparing with the general form $ax^2 + 2hxy + by^2 = 0$, we get
 $a = 6$, $h = \frac{-5}{2}$, $b = 1$.

Then, the pair of lines is given as follow:

$$y = \left(\frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 6}}{1} \right) x \rightarrow$$

$$y = \left(\frac{\frac{5}{2} \pm \sqrt{\frac{1}{4}}}{1} \right) x \rightarrow$$

$$y = \left(\frac{5}{2} + \frac{1}{2} \right) x \quad \& \quad y = \left(\frac{5}{2} - \frac{1}{2} \right) x \rightarrow$$

$$y = 3x \quad \& \quad y = 2x$$

Example (2.2): Let $8x^2 - 6xy + y^2 = 0$. Find the equation of the pair of lines.

Solution:

By comparing with the general form $ax^2 + 2hxy + by^2 = 0$, we get
 $a = 8$, $h = \frac{-6}{2} = -3$, $b = 1$.

Then, the pair of lines is given as follow:

$$y = \left(\frac{3 \pm \sqrt{9-8}}{1} \right) x \rightarrow$$

$$y = (3 \pm 1)x \rightarrow$$

$$y = 4x \quad \& \quad y = 2x.$$

Homework (2.1):

- (a) Let $x^2 + 3xy + 2y^2 = 0$. Find the equations of the pair of lines.
- (b) Let $3x^2 + 4xy + 2y^2 = 0$. Find the equations of the pair of lines.
- (c) Let $x^2 - 3xy + y^2 = 0$. Find the equations of the pair of lines.
- (d) Let $5x^2 + 6xy - 3y^2 = 0$. Find the equations of the pair of lines.
- (e) Let $7x^2 - xy - 2y^2 = 0$. Find the equations of the pair of lines.

Angle between a pair of lines:

The angle between the lines represented by the homogenous second degree equation $ax^2 + 2hxy + by^2 = 0$

is given by

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a+b}.$$

Example(2.3): Find the angle between lines represented by the homogenous equation $x^2 + 3xy + 2y^2 = 0$.

Solution:

$$a = 1, \quad h = \frac{3}{2}, \quad b = 2.$$

Then,

$$\tan \theta = \frac{2\sqrt{h^2-ab}}{a+b} = \frac{2\sqrt{\frac{9}{4}-2}}{3} = \frac{2\sqrt{\frac{1}{4}}}{3} = \frac{1}{3} \rightarrow$$

$$\theta = \tan^{-1}\left(\frac{1}{3}\right) = 18.4^\circ .$$

Example(2.4): Find the angle between the two lines represented by the homogenous equation: $3x^2 + 8xy + 4y^2 = 0$.

Solution:

$$a = 3 \quad , \quad h = \frac{8}{2} = 4 \quad , \quad b = 4 .$$

Then,

$$\tan \theta = \frac{2\sqrt{h^2-ab}}{a+b} = \frac{2\sqrt{16-12}}{7} = \frac{2\sqrt{4}}{7} = \frac{4}{7}$$

$$\theta = \tan^{-1}\left(\frac{4}{7}\right) = 29.7^\circ .$$

Homework (2.2):

- (a) Find the angle between the two lines represented by the homogenous equation $x^2 + 3xy - 2y^2 = 0$.
- (b) Find the angle between the two lines represented by the homogenous equation $3x^2 - 4y^2 = 0$.
- (c) Find the angle between the two lines represented by the homogenous equation $x^2 - 3xy - y^2 = 0$.
- (d) Find the angle between the two lines represented by the homogenous equation $x^2 + 2xy - 3y^2 = 0$.

Bisectors of a pair of lines:

For a given equation representing a pair of lines, the equation of bisectors of the angles between this pair of lines is given by the equation:

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}, \text{ and it easier to say the bisectors equation is given by:}$$

$$(a - b)xy = h(x^2 - y^2).$$

Example(2.5): Find the equation of the bisectors of the angle between the lines represented by $6x^2 - 13xy + 5y^2 = 0$.

Solution:

By comparing $ax^2 + 2hxy + by^2 = 0$ with $x^2 - 13xy + 5y^2 = 0$,

we get that

$$a = 6, \quad h = \frac{-13}{2}, \quad b = 5.$$

The bisectors equation is given by:

$$(a - b)xy = h(x^2 - y^2) \rightarrow$$

$$(6 - 5)xy = \frac{-13}{2}(x^2 - y^2) \rightarrow$$

$$-13(x^2 - y^2) = 2xy \rightarrow$$

$$-13x^2 + 13y^2 - 2xy = 0,$$

and multiplying this equation in (-1) and rearranging, we get

$$13x^2 + 2xy - 13y^2 = 0.$$

Example(2.6): Find the equation of the bisectors of the angle between the lines represented by $3x^2 + 8xy + 4y^2 = 0$.

Solution:

By comparing $ax^2 + 2hxy + by^2 = 0$ with $3x^2 + 8xy + 4y^2 = 0$, we get that

$$a = 3, \quad h = \frac{8}{2} = 4, \quad b = 4.$$

The bisectors equation is given by: $(a - b)xy = h(x^2 - y^2) \rightarrow$

$$(3 - 4)xy = 4(x^2 - y^2) \rightarrow$$

$$4(x^2 - y^2) = -xy \rightarrow$$

$$x^2 - 4y^2 + xy = 0,$$

and after rearranging, we get

$$4x^2 + xy - 4y^2 = 0.$$

Homework (2.3):

- (a) Find the equation of the bisectors of the angle between the lines represented by $12x^2 - 7xy - 4y^2 = 0$.
- (b) Find the angle between the two lines represented by the homogenous equation $3x^2 - 4y^2 = 0$.
- (c) Find the angle between the two lines represented by the homogenous equation $x^2 - 3xy - y^2 = 0$.
- (d) Find the angle between the two lines represented by the homogenous equation $x^2 + 2xy - 3y^2 = 0$.

(*) Now, the non-homogenous general equation of second degree

$$a x^2 + 2h x y + b y^2 + 2g x + 2f y + c = 0$$

represents a pair of lines if the following condition is satisfied:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, \text{ or equivalently}$$

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Note that: The two lines are perpendicular if $a + b = 0$.

Example(2.7): Show that the equation $2x^2 - 11xy + 5y^2 - x + 23y - 10 = 0$ represents a pair of lines.

Solution:

By comparing

$$a x^2 + 2h x y + b y^2 + 2g x + 2f y + c = 0 \quad \text{with} \\ 2x^2 - 11xy + 5y^2 - x + 23y - 10 = 0,$$

we get that:

$$a = 2, \quad b = 5, \quad c = -10, \quad f = 11.5, \quad g = -0.5, \quad h = -5.5.$$

Now, check the value $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, then

$$(2 \times 5 \times -10) + (2 \times 11.5 \times -0.5 \times -5.5) - (2 \times 11.5 \times 11.5) - \\ (5 \times -0.5 \times -0.5) - (-10 \times -5.5 \times -5.5) = 0.$$

Hence, the equation represents a pair of lines.

Example(2.8): Show if the equation $x^2 + 8xy + 16x + 4y + 4 = 0$ represents a pair of lines or not.

Solution:

By comparing

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{with} \\ x^2 + 8xy + 16x + 4y + 4 = 0,$$

we get that

$$a = 1, b = 0, c = 4, f = 2, g = 8, h = 4.$$

Now check the value $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, then

$$(1 \times 0 \times 4) + (2 \times 2 \times 8 \times 4) - (1 \times 2 \times 2) - (0 \times 8 \times 8) - (4 \times 4 \times 4) = \\ 128 - 4 - 64 = 128 - 68 = 60 \neq 0.$$

Hence, this equation does not represent a pair of lines.

Homework (2.4):

- (a) Show if the equation $3x^2 + 4xy + 8x + 2y + 14 = 0$ represents a pair of lines or not.
- (b) Show if the equation $y^2 - 5xy + 6x + 12y + 7 = 0$ represents a pair of lines or not.
- (c) Show if the equation $x^2 + 10x + 4y^2 + 24 = 0$ represents a pair of lines or not.

(*) If $h^2 - ab > 0$, then the two lines are real distinct lines and there is a point of intersection as follows:

$$(\alpha, \beta) = \left(\frac{bg-hf}{h^2-ab}, \frac{af-hg}{h^2-ab} \right).$$

Example(2.9): Show that the equation $3x^2 + 4xy + y^2 - 10x - 4y + 3 = 0$ represents a pair of lines and find its point of intersection.

Solution:

Since $a = 3$, $b = 1$, $c = 3$, $f = -2$, $g = -5$, $h = 2$.

Now check the value $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, then

$$(3 \times 1 \times 3) + (2 \times -2 \times -5 \times 2) - (3 \times -2 \times -2) - (1 \times -5 \times -5) - (3 \times 2 \times 2) = 49 - 12 - 25 - 12 = 49 - 49 = 0.$$

Hence, this equation represents a pair of lines, and since $h^2 - ab = 1 > 0$, then the two real lines of the pair intersect and the point of intersection is given by:

$$(\alpha, \beta) = \left(\frac{bg-hf}{h^2-ab}, \frac{af-hg}{h^2-ab} \right) = \left(\frac{-5+4}{1}, \frac{-6+10}{1} \right) = (-1, 4).$$

Axes transfer:

Let Ox, Oy be the original Cartesian coordinates that mean usually $-axis$, $Y - axis$. Suppose that these axis are transformed into new axis OX, OY . This transformation is coming from altering the Origin $O(0,0)$ to a new Origin $\acute{O}(a,b)$, so that any point P has old coordinates (x,y) and new coordinates (X,Y) , and the relation between old and new coordinates are:

$$x = X + a, \quad y = Y + b.$$

(In other words, it is called that the axis are translated to the point (a,b)).

Example(2.10): If the axes are translated to the point $(-4, -5)$. Find the new rotation of equation $x^2 + y^2 + 8x + 10y - 8 = 0$.

Solution:

$$x = X - 4 \quad , \quad y = Y - 5 \quad \rightarrow$$

$$(X - 4)^2 + (Y - 5)^2 + 8(X - 4) + 10(Y - 5) - 8 = 0 \quad \rightarrow$$

$$X^2 - 8X + 16 + Y^2 - 10Y + 25 + 8X - 32 + 10Y - 50 - 8 \quad \rightarrow$$

$$X^2 + Y^2 - 49 = 0 .$$

Example(2.11): If the axes are translated to the point $(-2, 3)$. Find the new rotation of equation $2x^2 + 4xy + 5y^2 - 4x - 22y + 7 = 0$.

Solution:

$$x = X - 2 \quad , \quad y = Y + 3 \quad \rightarrow$$

$$2(X - 2)^2 + 4(X - 2)(Y + 3) + 5(Y + 3)^2 - 4(X - 2) - 22(Y + 3) + 7 = 0 \quad \rightarrow$$

$$2X^2 + 4XY + 5Y^2 - 22 = 0 .$$

Homework (2.5):

- (a) If the axes are translated to the point $(-1, 2)$. Find the new rotation of equation $3 + 4xy + 6y^2 - 14x - 12y + 5 = 0$.
- (b) If the axes are translated to the point $(-2, 3)$. Find the new rotation of equation $y^2 - 5xy + 6x + 10y + 7 = 0$.
- (c) If the axes are translated to the point $(-2, 3)$. Find the new rotation of equation $x^2 + 10x + 4y^2 + 24 = 0$.

- (d) Show that: the equation $3x^2 - 2xy - y^2 + 8x + 4y - 3 = 0$ represents a pair of lines, and then show that the two lines intersect, find the angel between them and the point of intersection.
- (e) Find the value of μ so that the equation $3x^2 - 2xy - y^2 + 8x + 4y + \mu = 0$ represents a pair of lines, and then show that the two lines intersect, find the angel between them and the point of intersection.

B. The Circle:

(*) Standard equation of a circle:

To find the standard equation of a circle with a given center (h, k) and radius r , let (x, y) be any point on the circle. Then,

$$r = \sqrt{(x - h)^2 + (y - k)^2}$$

That means the standard equation of a circle with center (h, k) and radius r is given by

$$(x - h)^2 + (y - k)^2 = r^2$$

Example(2.12): If the center of the circle is $(3, -2)$ and the radius is 4, Find the standard equation of the circle.

Solution:

$$h = 3, \quad k = -2, \quad r = 4 \rightarrow$$

$$(x - 3)^2 + (y - (-2))^2 = 4^2 \rightarrow$$

$$(x - 3)^2 + (y + 2)^2 = 4^2 \rightarrow$$

$$x^2 - 6x + 9 + y^2 + 4y + 4 = 16 \rightarrow$$

$$x^2 - 6x + y^2 + 4y + 13 = 16 \rightarrow$$

$$x^2 + y^2 - 6x + 4y - 3 = 0.$$

Example(2.13): If the center of the circle is $(1, 2)$ and the radius is 4, Find the standard equation of the circle.

Solution:

$$h = 1, \quad k = 2, \quad r = 4 \rightarrow$$

$$(x - 1)^2 + (y - 2)^2 = 4^2 \rightarrow$$

$$(x - 1)^2 + (y - 2)^2 = 16 \rightarrow$$

$$x^2 - 2x + 1 + y^2 - 4y + 4 = 16 \rightarrow$$

$$x^2 - 2x + y^2 - 4y + 5 = 16 \rightarrow$$

$$x^2 + y^2 - 2x - 4y - 11 = 0.$$

(*) The general equation of second degree that represents a circle is:

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

where $(-g, -f)$ is the center of the circle

and $r = \sqrt{g^2 + f^2 - c}$ gives us the radius of this circle.

Note that: In this special general equation, we have $a = b$, $h = 0$, and $\{f, g, c\}$ are computed after setting the coefficients of x^2 and y^2 equal to 1 .

From the general equation of a circle, we can find the center and the radius of the circle as in before.

Example(2.14): Find the center and the radius of the circle:

$$x^2 + y^2 + 4x + 8y - 41 = 0 .$$

Solution:

$$g = 2 , f = 4 , c = -41 .$$

The center is $(-g, -f) \rightarrow (-2, -4)$,

and the radius is $r = \sqrt{g^2 + f^2 - c} = \sqrt{(2)^2 + (4)^2 - (-41)} = \sqrt{61}$.

Example(2.15): Find the center and the radius of the circle:

$$x^2 + y^2 - 4x - 16 = 0 .$$

Solution:

$$g = -2 , f = 0 , c = -16 .$$

The center is $(-g, -f) = (2, 0)$,

and the radius is

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{(2)^2 + (0)^2 - (-16)} = \sqrt{20} = 2\sqrt{5} .$$

Note that:

If $g^2 + f^2 - c > 0$, then there is a real circle with a radius $r > 0$.

If $g^2 + f^2 - c = 0$, then the circle is real but it is just a point in the plane coincides with the center and the radius is zero.

If $g^2 + f^2 - c < 0$, then the circle is imaginary not real.

Example(2.16): Change the equation $4x^2 + 4y^2 - 12x + 4y = 26$ to the standard form.

Solution:

First, we must divide the equation over , then

$$x^2 + y^2 - 3x + y - \frac{26}{4} = 0 .$$

$$g = -\frac{3}{2} , \quad f = \frac{1}{2} , \quad c = -\frac{26}{4} .$$

The center is $(-g, -f) = (\frac{3}{2}, -\frac{1}{2})$, the radius is

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{9}{4} + \frac{1}{4} + \frac{26}{4}} = \sqrt{9} = 3 .$$

Hence, the standard form is

$$\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 9 .$$

(*) Circles with a given diameter:

To find the equation of a circle using the end points $A(x_1, y_1)$, $B(x_2, y_2)$ of a diameter in that circle, let $P(x, y)$ be any point of the circle. Connecting the point $A(x_1, y_1)$ and the point $B(x_2, y_2)$ with the point P which makes an angle 90° between PA , PB , we find the slopes of the lines PA and PB

The slope of the line PA is $m = \frac{y-y_1}{x-x_1}$

The slope of the line PB is $m' = \frac{y-y_2}{x-x_2}$.

Since the angle $APB = 90^\circ$, then the lines PA and PB are perpendicular to each other and therefore, $mm' = -1$.

The equation of the circle through the extremities (end points) of its diameter is:

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1 \rightarrow$$

$$(y-y_1)(y-y_2) = -(x-x_1)(x-x_2) \rightarrow$$

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0 \dots\dots\dots (*) .$$

Also, the equation of the circle through the extremities (end points) of its diameter may be as follows:

$$x^2 - x x_1 - x x_2 + x_1 x_2 + y^2 - y y_1 - y y_2 + y_1 y_2 = 0 \rightarrow$$

$$x^2 + y^2 - (x_1+x_2) x - (y_1 + y_2) y + x_1 x_2 + y_1 y_2 = 0 \dots\dots\dots (**) .$$

Comparing this equation with the general equation of a circle

$$x^2 + y^2 - (x_1+x_2)x - (y_1 + y_2)y + x_1 x_2 + y_1 y_2 = 0 \quad \text{and}$$

$$x^2 + y^2 + 2gx + 2fy + c = 0, \text{ then}$$

$$g = -\frac{(x_1+x_2)}{2}, \quad f = -\frac{(y_1+y_2)}{2}, \quad c = x_1 x_2 + y_1 y_2.$$

The center is given by

$$(h, k) = (-g, -f) = \left(\frac{(x_1+x_2)}{2}, \frac{(y_1+y_2)}{2} \right).$$

The radius is given by

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{x_1+x_2}{2}\right)^2 + \left(-\frac{y_1+y_2}{2}\right)^2 - (x_1 x_2 + y_1 y_2)}.$$

Example(2.17): Find the equation of a circle through the ends $(8, -2)$ and $(2, -6)$ of its diameter . Also find the center and radius.

Solution :

$$x_1 = 8, \quad y_1 = -2, \quad x_2 = 2, \quad y_2 = -6 \rightarrow$$

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0 \rightarrow$$

$$(x - 8)(x - 2) + (y - (-2))(y - (-6)) = 0 \rightarrow$$

$$(x - 8)(x - 2) + (y + 2)(y + 6) = 0 \rightarrow$$

$$x^2 - 10x + 16 + y^2 + 8y + 12 = 0 \rightarrow$$

$$x^2 + y^2 - 10x + 8y + 28 = 0.$$

Remark:

$$(-g, -f) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left(\frac{10}{2}, -\frac{8}{2} \right) = (5, -4),$$

$$r = \sqrt{\left(-\frac{x_1 + x_2}{2} \right)^2 + \left(-\frac{y_1 + y_2}{2} \right)^2 - (x_1 x_2 + y_1 y_2)} \rightarrow$$

$$r = \sqrt{(5)^2 + (-4)^2 - (16 + 12)} = \sqrt{13}.$$

Example(2.18): Find the equation of a circle through the end points (5, 1) and (7, 5) of its diameter. Also, find the center and the radius.

Solution:

$$(x_1, y_1) = (5, 1), (x_2, y_2) = (7, 5).$$

Then, the equation of the Circle:

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0 \rightarrow$$

$$(x - 5)(x - 7) + (y - 1)(y - 5) = 0 \rightarrow$$

$$x^2 - 12x + 35 + y^2 - 6y + 5 = 0 \rightarrow$$

$$x^2 - 12x + 35 + y^2 - 6y + 5 = 0 \rightarrow$$

$$x^2 + y^2 - 12x - 6y + 40 = 0.$$

The center and the radius are given by

$$(-g, -f) = \left(\frac{12}{2}, \frac{6}{2} \right) = (6, 3) \quad \& \quad r = \sqrt{g^2 + f^2 - c} = \sqrt{6^2 + 3^2 - 40} = \sqrt{5}.$$

(*) Equation of the tangent at a point on a circle:

The equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) is:

$$x x_1 + y y_1 + g (x + x_1) + f (y + y_1) + c = 0$$

Example(2.19): Find the equation of the tangent to the circle $x^2 + y^2 - 4y - 1 = 0$ at the point $(2,1)$.

Solution:

By comparing

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{and}$$

$$x^2 + y^2 - 4y - 1 = 0 ,$$

we get

$$2g = 0, \quad 2f = -4 \rightarrow g = 0, \quad f = -2, \quad c = -1, \quad x_1 = 2, \quad y_1 = 1 .$$

Then, the equation of the tangent to the circle is

$$x x_1 + y y_1 + g (x + x_1) + f (y + y_1) + c = 0 \rightarrow$$

$$x(2) + y(1) + (0)(x + 2) + (-2)(y + 1) + (-1) = 0 \rightarrow$$

$$2x + y - 2y - 2 - 1 = 0 \rightarrow$$

$$2x - y - 3 = 0 .$$

(*) Length of the tangent:

The length L of the tangent to a given circle at the point (x_1, y_1) is given directly from:

$$L = \sqrt{x_1^2 + y_1^2 + 2g x_1 + 2f y_1 + c} .$$

Example(2.20): Find the length of the tangent to a circle $x^2 + y^2 - 5x + 3 = 0$ at the point (2, 4).

Solution:

$$g = -\frac{5}{2} , f = 0 , c = 3 , x_1 = 2 , y_1 = 4 .$$

Then, the length L of the tangent to the circle is given by

$$L = \sqrt{x_1^2 + y_1^2 + 2g x_1 + 2f y_1 + c} \rightarrow$$

$$L = \sqrt{4 + 16 + (-5)(2) + (0)(4) + (3)} = \sqrt{13} .$$

Example(2.21): Find the length of the tangent drawn from the point $(-3, 4)$ to the circle $x^2 + y^2 - 4x - 6y - 3 = 0$.

Solution:

$$g = -2 , f = -3 , c = -3 , x_1 = -3 , y_1 = 4 .$$

Then, the length L of the tangent to the circle is given by

$$L = \sqrt{x_1^2 + y_1^2 + 2g x_1 + 2f y_1 + c} \rightarrow$$

$$L = \sqrt{9 + 16 + (-4)(-3) + (-6)(4) + (-3)} = \sqrt{10} .$$

(*) The common line between two circles:

The equation of the common line between two circles

$$x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0 \quad \text{and}$$

$$x^2 + y^2 + 2g_2 x + 2f_2 y + c_2 = 0$$

is given directly by the following equation:

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0.$$

Example(2.22): Find the common line between the circles:

$$x^2 + y^2 + 6x + 8y + 3 = 0 \quad \text{and} \quad x^2 + y^2 + 4x - 2y - 5 = 0.$$

Solution:

$$g_1 = 3, f_1 = 4, c_1 = 3, \quad g_2 = 2, f_2 = -1, c_2 = -5.$$

Then,

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0 \rightarrow$$

$$2(3 - 2)x + 2(4 + 1)y + (3 + 5) = 0 \rightarrow$$

$$2x + 6y + 8 = 0.$$

(*) Orthogonal circles:

The condition so that the two circles

$$x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0 \quad \text{and}$$

$$x^2 + y^2 + 2g_2 x + 2f_2 y + c_2 = 0$$

is given directly from the following:

$$2 g_1 g_2 + 2 f_1 f_2 = c_1 + c_2 .$$

Example(2.23): Prove that the circles:

$2x^2 + 2y^2 - 7x + 5 = 0$ and $x^2 + y^2 - 6x + 4y + 8 = 0$
are orthogonal.

Solution:

$$g_1 = -\frac{7}{4} , f_1 = 0 , c_1 = \frac{5}{2} , \quad g_2 = -3, f_2 = 2 , c_2 = 8 .$$

Then,

$$2 g_1 g_2 + 2 f_1 f_2 = (2) \left(-\frac{7}{4}\right) (-3) + (2)(0)(2) = \frac{21}{2} ,$$

$$c_1 + c_2 = \frac{5}{2} + 8 = \frac{21}{2} .$$

Hence, $2 g_1 g_2 + 2 f_1 f_2 = c_1 + c_2$, and the circles are orthogonal.

Homework (2.6):

(a) Find the equation of tangent to the circle $x^2 + y^2 = 13$ at the point $(2, 3)$.

(b) If the axes are translated to the point $(4, -1)$. Find the new rotation of the equation $3x^2 - y^2 - 2xy + 6x + 14y + 8 = 0$.

(c) Change the equation $3x^2 + 3y^2 - 9x + 12y + 6 = 0$ to the standard form.

(d) Find the center and the radius of the circle:

$$5x^2 + 5y^2 - 10x + 20y + 15 = 0.$$

(e) Find the equation of the circle whose center is $(-3, -5)$ and radius is .

(f) Find the equation of the circle whose diameter is the line joining the points $(-2, 4)$, $(6, 0)$.

(g) Find the common line between the circles:

$$x^2 + y^2 + 6x + 8y + 3 = 0 \quad \text{and} \quad x^2 + y^2 + 4x - 2y - 5 = 0.$$

(h) Find the equation of the curve $x^2 + y^2 - 6x - 10y - 2 = 0$ if the origin moved into $(3, 5)$.

(*) The equation of a circle passing through three points:

The general equation of a circle is: $x^2 + y^2 + 2gx + 2fy + c = 0$.

If we ask for the equation of a circle passing through three points (a, b) , (r, s) , (u, v) , then these points satisfy the equation of the circle. That is, replacing x with a and replacing y with b , we get the first equation:

$$a^2 + b^2 + 2g(a) + 2f(b) + c = 0 \rightarrow (1)$$

Similarly, we get that:

$$r^2 + s^2 + 2g(r) + 2f(s) + c = 0 \rightarrow (2)$$

$$u^2 + v^2 + 2g(u) + 2f(v) + c = 0 \rightarrow (3)$$

Solving these equations, we get the values of g , f , c of the circle, and then find the equation of the circle.

Example(2.24): Find the equation of the circle passing through the three points: $(-2, 1)$, $(-2, -3)$, $(4, -3)$.

Solution:

The general equation is $x^2 + y^2 + 2gx + 2fy + c = 0$, and then

$$(-2)^2 + (1)^2 + 2g(-2) + 2f(1) + c = 0 \rightarrow (*),$$

$$(-2)^2 + (-3)^2 + 2g(-2) + 2f(-3) + c = 0 \rightarrow (**),$$

$$(4)^2 + (-3)^2 + 2g(4) + 2f(-3) + c = 0 \rightarrow (***) .$$

That is,

$$-4g + 2f + c = -5 \rightarrow (1),$$

$$-4g - 6f + c = -13 \rightarrow (2),$$

$$8g - 6f + c = -25 \rightarrow (3) .$$

By subtracting (1) , (2):

$$8f = 8 \rightarrow f = 1 \rightarrow (4)$$

By subtracting (3) , (2):

$$12g = -12 \rightarrow g = -1 \rightarrow (5)$$

Substituting (4) , (5) in (1):

$$4 + 2 + c = -5 \rightarrow c = -11 .$$

Now the equation of the circle is: $x^2 + y^2 + -2x + 2y - 11 = 0 .$

(*) The position of a line in relative to a circle:

For a circle of radius r and center (α , β) , and a line $ax + by + c = 0$, how to explain the position of the line relative to the circle.

If we found the perpendicular distance from the origin of the circle (α , β) to the line $ax + by + c = 0$, as we know in before by the relation:

$$d = \left| \frac{a(\alpha) + b(\beta) + c}{\sqrt{a^2 + b^2}} \right| , \quad \text{then we have three cases:}$$

- 1- $d < r$, then the line intersects the circle in two points.
- 2- $d = r$, then the line touches the circle in one point.
- 3- $d > r$, then the line is going outside the circle and not intersecting it.

Example(2.25): Show that: the line $2x + y = 11$ intersects the circle $x^2 + y^2 = 26$, and find the points of intersection.

Solution:

Here, we have the radius is $r = \sqrt{26}$ and the center is $(0, 0)$. Now, the distance from the center to the line $2x + y = 11$ is given by:

$$d = \left| \frac{a(\alpha) + b(\beta) + c}{\sqrt{a^2 + b^2}} \right| = \left| \frac{2(0) + 1(0) + (-11)}{\sqrt{4+1}} \right| = \frac{11}{\sqrt{5}} .$$

Now, since $\frac{11}{5} < \sqrt{26}$, then the line intersects the circle in two points.

To find these points of intersection, we solve the equations $2x + y = 11$ and $x^2 + y^2 = 26$ together.

If we take $y = 11 - 2x$ and put it in the circle, then

$$x^2 + (11 - 2x)^2 = 26 \rightarrow x^2 + 4x^2 + 121 - 44x - 26 = 0 \rightarrow$$

$$5x^2 - 44x + 95 = 0 \rightarrow (x - 5)(5x - 19) = 0 \rightarrow$$

$$x = 5 \text{ or } x = \frac{19}{5}.$$

Again from $y = 11 - 2x$, we get that the two points are $(5, 1)$, $(\frac{19}{5}, \frac{17}{5})$.

Homework (2.7): Is the line $3x + 4y - 2 = 0$ intersects the circle $x^2 + y^2 + 8x + 12y + 11 = 0$.

Chapter (3): Conic sections

(*) Second Degree Curve:

In Chapter (2), we get that the general equation of degree 2

$$a x^2 + 2h x y + b y^2 + 2g x + 2f y + c = 0$$

represents a pair of lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$,

and represents a circle if $a = b = 1$, $h = 0$.

Now, the general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents one of three geometric curves depending on the coefficients of that general equation. These three curves are called Conic Sections.

If the condition of being a pair of lines failed, that is,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0, \text{ and it is not a circle (} a \neq b, h \neq 0 \text{).}$$

Then, the general equation of degree 2 represents one of the following conic sections:

- (1) If $h^2 < ab \Rightarrow$ *The curve is an ellips*
- (2) If $h^2 = ab \Rightarrow$ *The curve is a Parabola*
- (3) If $h^2 > ab \Rightarrow$ *The curve is a hyperbola*

Example(3.1): Find the type of a curve if $y^2 - y - 2x + 4 = 0$.

Solution:

By comparing the given equation with the general equation,

$$a x^2 + 2h xy + b y^2 + 2g x + 2f y + c = 0$$

$$y^2 - y - 2x + 4 = 0 ,$$

we get $a = 0$, $b = 1$, $h = 0$, $g = -1$, $f = -\frac{1}{2}$, $c = 4$.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 4 \end{vmatrix} =$$

$$(0) \begin{vmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 4 \end{vmatrix} - (0) \begin{vmatrix} 0 & -\frac{1}{2} \\ -1 & 4 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 1 \\ -1 & -\frac{1}{2} \end{vmatrix} =$$

$$0 - 0 + (-1)(0 - (-1)) = -1 \neq 0 .$$

Now, we can determine the type of the curve where

$h = 0 \rightarrow h^2 = 0$, and $a b = (0)(1) = 0$, then

$h^2 = a b \Rightarrow$ the curve is a Parabola .

Example(3.2): Find the type of the curve if $x^2 + y^2 - 2x + 4y - 4 = 0$.

Solution:

By comparing the given equation with the general equation,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 ,$$

$$x^2 + y^2 - 2x + 4y - 4 = 0 ,$$

we get $a = 1$, $b = 1$, $h = 0$, $g = -1$, $f = 2$, $c = -4$,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -4 \end{vmatrix} =$$

$$1(-4 - 4) - (0)(0 - (-2)) + (-1)(0 - (-1)) = -9 \neq 0 .$$

Now, we can determine the type of the curve where

$$h = 0 \rightarrow h^2 = 0 \quad \text{and} \quad ab = (1)(1) = 1 , \text{ then}$$

$$h^2 < ab \Rightarrow \text{the curve is an ellipse} .$$

A. The Parabola

The Focus of a point p which moves so that its distance from a fixed point s (Focus) is always in a constant ratio (eccentricity) to its perpendicular distance from a point m on a fixed line AB (Directrix) is called a conic section.

Types of conic sections:

There are three types of conic section

$$\frac{ps}{pm} = \begin{cases} e = 1 & \text{Parabola} \\ e < 1 & \text{Ellipse} \\ e > 1 & \text{Hyperbola} \end{cases}$$

First: The parabola (eccentricity $e = 1$). There are four cases of parabola with vertex at the origin.

- 1) The equation of a parabola with vertex at the origin and focus at $(a, 0)$ is:

$$y^2 = 4ax, \quad a > 0.$$

It is concave right and the directrix equation is $x = -a$.

- 2) The equation of a parabola with vertex at the origin and focus at $(a, 0)$ is:

$$y^2 = 4ax, \quad a < 0.$$

It is concave left and the directrix equation is $x = -a$.

- 3) The equation of a parabola with vertex at the origin and focus at $(0, b)$ is:

$$x^2 = 4by, \quad b > 0.$$

It is concave up and the directrix equation is $y = -b$.

- 4) The equation of a parabola with vertex at the origin and focus at $(0, b)$ is:

$$x^2 = 4b y , b < 0 .$$

It is concave down and the directrix equation is $y = -b$.

For the more general case of parabola with vertex $v(h, k)$. There are four cases of parabola with a vertex at (h, k) .

- 1) The equation of a parabola with vertex v at (h, k) and focus s at $(h + a, k)$ is:

$$(y - k)^2 = 4a (x - h) , a > 0 .$$

It is concave right and the directrix equation is $x = h - a$.

- 2) The equation of a parabola with vertex v at (h, k) and focus s at $(h + a, k)$ is:

$$(y - k)^2 = 4a (x - h) , a < 0 .$$

It is concave left and the directrix equation is $x = h - a$.

- 3) The equation of a parabola with vertex v at (h, k) and focus s at $(h, k + b)$ is:

$$(x - h)^2 = 4b (y - k) , b > 0 .$$

It is concave up and the directrix equation is $y = k - b$.

- 4) The equation of a parabola with vertex v at (h, k) and focus s at $(h, k + b)$ is:

$$(x - h)^2 = 4b (y - k) , b < 0 .$$

It is concave down and the directrix equation is $y = k - b$.

These four equations are called the standard forms of any parabola.

Example(3.3): Find the focus and the directrix of the parabola $4x^2 = 36y$.

Solution:

$$4x^2 = 36y$$

$$\Rightarrow \frac{4x^2}{4} = \frac{36y}{4}$$

$$\Rightarrow x^2 = 9y$$

$$\Rightarrow x^2 = 4b y \Rightarrow 4b = 9 \Rightarrow b = \frac{9}{4} > 0 .$$

$\therefore$ The focus s at $(h, k + b)$ i.e. $s(0, \frac{9}{4})$ and the directrix is $y = k - b$
i.e. $y = -\frac{9}{4}$, and the parabola is concave up.

Example(3.4): Find the focus and the directrix of the parabola $(x + 2)^2 = -4(y + 1)$.

Solution:

$$(x + 2)^2 = -4(y + 1)$$

$$\Rightarrow (x - (-2))^2 = -4(y - (-1)) , \quad 4b = -4$$

$$\Rightarrow b = -1 < 0 , \quad h = -2 , \quad k = -1 .$$

$\therefore$ The focus s at $(h, k + b)$ i.e. $s(-2, -2)$ and the directrix is $y = k - b$
i.e. $y = 0$, and the parabola is concave down.

Example(3.5): Find the equation of the parabola with vertex $v (-1, -3)$ and focus $s (\frac{1}{4}, -3)$.

Solution:

Since s and v have the same y - *coordinates* , then the equation of the form $(y - k)^2 = 4a (x - h)$,

where $h = -1$, $k = -3$, $h + a = \frac{1}{4} \rightarrow a = \frac{1}{4} - (-1) = \frac{9}{4} > 0$.

$\therefore$ the equation is $(y - k)^2 = 4a (x - h) \rightarrow (y + 3)^2 = 4 \times \frac{9}{4} (x + 1) \rightarrow$

The standard form equation is $(y + 3)^2 = 9 (x + 1)$.

If we want the general form equation, then we have

$$y^2 + 6y + 9 = 9x + 9 \rightarrow$$

$$y^2 - 9x + 6y = 0 .$$

Example(3.6): Find the equation of the parabola with vertex $v (1, -3)$ and focus $s (1, 2)$.

Solution:

Since s and v have the same x - *coordinates* , then the equation of the form $(x - h)^2 = 4b (y - k)$,

where $h = 1$, $k = -3$, $k + b = 2 \rightarrow b = 2 - (-3) = 5 > 0$.

$\therefore$ the equation is $(x - h)^2 = 4b (y - k) \rightarrow (x - 1)^2 = 4 \times 5 (y + 3) \rightarrow$

The standard form equation is $(x - 1)^2 = 20 (y + 3)$.

If we want the general form equation, then we have

$$x^2 - 2x + 1 = 20y + 60 \rightarrow$$

$$x^2 - 2x - 20y - 59 = 0 .$$

Example(3.7): Sketch the parabola $x^2 - 4x + 4y = 0$.

Solution:

Completing the square in the variable x

(by adding and subtracting $\frac{(\text{cooefecient of } x)^2}{4}$), we get that

$$x^2 - 4x + 4 - 4 + 4y = 0 \rightarrow (x - 2)^2 + 4(y - 1) = 0 \rightarrow$$

The standard equation is $(x - 2)^2 = 4(-1)(y - 1)$, $b = -1 < 0$.

The parabola has vertex $v(h, k) = (2, 1)$ and focus $s(h, k + b) = (2, 0)$, and the directrix is $y = k - b \rightarrow y = 1 - (-1) = 2$.

Example(3.8): Sketch the parabola $x^2 + 4x - 4y = 0$.

Solution: In class.

Example(3.9): Sketch the parabola $y^2 + 2y - 4x + 5 = 0$.

Solution: In class.

Example(3.10): Sketch the parabola $y^2 - 4y - 4x + 8 = 0$.

Solution: In class.

Example(3.11): Sketch the parabola $x^2 - 2x - \frac{5}{2}y - \frac{3}{2} = 0$.

Solution: In class.

Homework(3.1):

- 1) Find the standard equation of the parabola with vertex $v(2, -3)$ and focus $s(-2, -3)$.
- 2) Find the general equation of the parabola with vertex $v(1, 5)$ and focus $s(-2, 5)$.
- 3) Find the standard equation of the parabola with vertex $v(1, -1)$ and focus $s(1, -4)$.
- 4) Find the general equation of the parabola with vertex $v(2, 2)$ and focus $s(-3, 2)$.
- 5) Find the focus and the directrix of the parabola $(y - 3)^2 = 8(x + 5)$.
- 6) Sketch the parabola $2y^2 - 5x - 4y - 3 = 0$.

B. The Ellipse:

The most important sign that the general equation represents a hyperbola is that $a, b > 0$ together or $a, b < 0$ together.

The equation of an ellipse with center at the origin $c(0, 0)$ is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0.$$

Properties of an ellipse with center $c(0, 0)$:

Equations of Ellipse	Horizontal $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$	Vertical $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, b > a$
x – intercepts	$\pm a$	$\pm a$
y – intercepts	$\pm b$	$\pm b$
Major axes Length	$2a$	$2b$
Minor axes Length	$2b$	$2a$
Eccentricity	$e = \frac{\sqrt{a^2 - b^2}}{a} < 1, ae = m = \sqrt{a^2 - b^2}$	$e = \frac{\sqrt{b^2 - a^2}}{b} < 1, be = m = \sqrt{b^2 - a^2}$
vertices	$(\pm a, 0)$ on major axes , $(0, \pm b)$ on minor axes	$(0, \pm b)$ on major axes , $(\pm a, 0)$ on minor axes
Foci	$s_1(m, 0), s_2(-m, 0)$	$s_1(0, m), s_2(0, -m)$
Directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$

The equation of an ellipse with center at any point $c(h, k)$ is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a, b > 0.$$

Properties of an ellipse with $c(h, k)$:

Equations of Ellipse	Horizontal $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, a > b$	Vertical $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, b > a$
x – intercepts	$h \pm a$	$h \pm a$
y – intercepts	$k \pm b$	$k \pm b$
Major axes Length	$2a$	$2b$
Minor axes Length	$2b$	$2a$
Eccentricity	$e = \frac{\sqrt{a^2 - b^2}}{a} < 1, ae = m = \sqrt{a^2 - b^2}$	$e = \frac{\sqrt{b^2 - a^2}}{b} < 1, be = m = \sqrt{b^2 - a^2}$
Vertices	$(h \pm a, k)$ on major axes , $(h, k \pm b)$ on minor axes	$(h, k \pm b)$ on major axes , $(h \pm a, k)$ on minor axes
Foci	$s_1(h + m, k), s_2(h - m, k)$	$s_1(h, k + m), s_2(h, k - m)$
Directrices	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{b}{e}$

Example(3.12): Find the *major and minor vertices* , *foci* of the ellipse $4x^2 + 9y^2 = 36$.

Solution:

$$\frac{4x^2}{36} + \frac{9y^2}{36} = \frac{36}{36} = 1 \rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$

is a horizontal ellipse with center $c(0, 0)$.

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \rightarrow a^2 = 9 \rightarrow a = 3, \quad b^2 = 4 \rightarrow b = 2 \quad \text{and } a > b ,$$

Eccentricity	$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{9 - 4}}{3} = \frac{\sqrt{5}}{3} < 1 , \quad ae = \sqrt{5}$
Directrices	$x = \pm \frac{a}{e} = \pm \frac{3}{\sqrt{5}/3} = \pm \frac{9}{\sqrt{5}}$

and then

$$\text{major vertices} = x - \text{intercept} = \pm 3 ,$$

$$\text{minor vertices} = y - \text{intercept} = \pm 2 ,$$

$$\text{and the foci are } s_1 (ae, 0) = (\sqrt{5}, 0) , \quad s_2 (-ae, 0) = (-\sqrt{5}, 0) ,$$

$$\text{and the major vertices are } (\pm a, 0) = (\pm 3, 0)$$

$$\text{and the minor vertices are } (0, \pm b) = (0, \pm 2) .$$

Example(3.13): Find the major length and the minor length of the ellipse $4x^2 + y^2 = 16$

Solution: For $4x^2 + y^2 = 16$, we have

$$\frac{4x^2}{16} + \frac{y^2}{16} = \frac{16}{16} = 1 \rightarrow$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 1, \quad a^2 = 4 \rightarrow a = 2, \quad b^2 = 16 \rightarrow b = 4 \text{ and } a < b.$$

It is a vertical ellipse with center $c(0, 0)$ and then

the eccentricity is $e = \frac{\sqrt{b^2 - a^2}}{b} = \frac{\sqrt{16 - 4}}{4} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2} < 1, \quad be = 2\sqrt{3}.$

the major length $= 2b = 2(4) = 8,$
the minor length $= 2a = 2(2) = 4.$

Directrices	$y = \pm \frac{b}{e} = \pm \frac{4}{\sqrt{3}/2} \rightarrow y = \pm \frac{8}{\sqrt{3}}$
-------------	---

The foci are $s_1(0, be) = (0, 2\sqrt{3})$, $s_2(0, -be) = (0, -2\sqrt{3}),$

and the major vertices are $(0, \pm b) = (0, \pm 4)$
 and the minor vertices are $(\pm a, 0) = (\pm 2, 0).$

Example(3.14): Find the center, the foci, the major and minor vertices and the directrices of the ellipse $4x^2 + 9y^2 - 48x + 72y + 144 = 0.$

Solution: Since the equation is also: $4(x^2 - 12x) + 9(y^2 + 8y) + 144 = 0.$

Then completing the square in the variables x and y by

adding and subtracting $\frac{(\text{coefficient of } x)^2}{4}$ and $\frac{(\text{coefficient of } y)^2}{4}$, we get that

$$4(x^2 - 12x + 36 - 36) + 9(y^2 + 8y + 16 - 16) + 144 = 0 \rightarrow$$

$$4(x - 6)^2 + 9(y + 4)^2 - 144 - 144 + 144 = 0 \rightarrow$$

$$4(x - 6)^2 + 9(y - (-4))^2 = 144 \rightarrow$$

$$\frac{(x-6)^2}{36} + \frac{(y-(-4))^2}{16} = 1 .$$

It is a horizontal ellipse with center $c(h, k) = (6, -4)$, $a = 6$, $b = 4$, $a > b$,

the eccentricity $e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{20}}{6} = \frac{\sqrt{5}}{3} < 1 \rightarrow$

$$\pm ae = \pm 2\sqrt{5} \quad , \quad \pm \frac{a}{e} = \pm \frac{18}{\sqrt{5}} .$$

The foci are $s_1(h + ae, k) = (6 + 2\sqrt{5}, -4)$, $s_2(h - ae, k) = (6 - 2\sqrt{5}, -4)$,

the major vertices are $(h \pm a, k) = (12, -4)$, $(0, -4)$ and the minor vertices are $(h, k \pm b) = (6, 0)$, $(6, -8)$.

The directrices are $x = h \pm \frac{a}{e} = 6 \pm \frac{18}{\sqrt{5}}$, and the major length is $2a = 12$ and the minor length is $2b = 8$.

Homework (3.2):

- 1- Find the equation of the ellipse with foci $(4, 5)$, $(0, 5)$ and vertices $(-3, 5)$, $(7, 5)$.
- 2- Find the equation of the ellipse with foci $(1, 4)$, $(1, 5 - 4)$ and major length 10.

C. The hyperbola:

The most important sign that the general equation represents a hyperbola is that *either* $(a > 0, b < 0)$ *or* $(a < 0, b > 0)$.

The equation of a hyperbola with center $A(0, 0)$ is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

where

$e > 1$, $ae = m$ (horizontal hyperbola), $be = m$ (vertical hyperbola).

Properties of Hyperbola with center $c(0, 0)$:

equation of hyperbola	Horizontal $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Vertical $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Foci	$(\pm m, 0)$, $m^2 = a^2 + b^2$	$(0, \pm m)$, $m^2 = a^2 + b^2$
Asymptotes	$y = \pm \frac{a}{b}x$	$y = \pm \frac{b}{a}x$
Eccentricity	$e = \frac{\sqrt{a^2+b^2}}{a} = \frac{m}{a} > 1$, $ae = m$	$e = \frac{\sqrt{a^2+b^2}}{b} = \frac{m}{b} > 1$, $be = m$
Directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$

Properties of Hyperbola with center $c(h, k)$:

Equation of hyperbola	Horizontal $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	Vertical $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
Vertices	$(h \pm a, k)$	$(h, k \pm b)$
Foci	$(h \pm m, k)$, $m^2 = a^2 + b^2$	$(h, k \pm m)$, $m^2 = a^2 + b^2$
Asymptotes	$y = \pm \frac{a}{b}x$	$y = \pm \frac{b}{a}x$
Eccentricity	$e = \frac{\sqrt{a^2+b^2}}{a} = \frac{m}{a} > 1$, $ae = m$	$e = \frac{\sqrt{a^2+b^2}}{b} = \frac{m}{b} > 1$, $be = m$
Directrices	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{b}{e}$

Example(3.15): Find the center, the foci, the asymptotes, the vertices and the directrices of the hyperbola $4x^2 - y^2 = 36$.

Solution:

$$4x^2 - y^2 = 36 \rightarrow$$

$$\frac{4x^2}{36} - \frac{y^2}{36} = \frac{36}{36} \rightarrow$$

$$\frac{x^2}{9} - \frac{y^2}{36} = 1$$

and then it is a horizontal hyperbola of center $c(0, 0)$.

$$a^2 = 9 \rightarrow a = 3, \quad b^2 = 36 \rightarrow b = 6,$$

$$m^2 = 9 + 36 = 45 \rightarrow m = \sqrt{45} = 3\sqrt{5}, \quad m = a e \rightarrow e = \sqrt{5} > 1.$$

vertices	$(\pm a, 0) = (\pm 3, 0)$, vertices are $(3, 0), (-3, 0)$
foci	$(\pm m, 0) = (\pm 3\sqrt{5}, 0)$, foci are $(+3\sqrt{5}, 0), (-3\sqrt{5}, 0)$
Asymptotes	$y = \pm \frac{a}{b}x \rightarrow y = \pm \frac{1}{2}x \rightarrow$ $y = \frac{1}{2}x, \quad y = -\frac{1}{2}x$
Directices	$x = \pm \frac{a}{e} \rightarrow x = \pm \frac{3}{\sqrt{5}}$

Example(3.16): Find the center, the foci, the asymptotes, the vertices and the directrices of the hyperbola $9y^2 - 16x^2 = 144$.

Solution:

$$9y^2 - 16x^2 = 144 \rightarrow$$

$$\frac{9y^2}{144} - \frac{16x^2}{144} = \frac{144}{144} = 1 \rightarrow$$

$$\frac{y^2}{16} - \frac{x^2}{9} = 1.$$

Then, it is a vertical hyperbola with center $c(0, 0)$ and

$$b^2 = 16 \rightarrow b = 4, \quad a^2 = 9 \rightarrow a = 3$$

$$m^2 = 16 + 9 = 25 \rightarrow m = \sqrt{25} = 5, \quad be = m \rightarrow e = \frac{5}{4} = 1.25 > 1.$$

vertices	$(0, \pm b) = (0, \pm 4)$, vertices are $(0, 4), (0, -4)$
foci	$(0, \pm m) = (0, \pm 5) \rightarrow (0, 5), (0, -5)$
Asymptotes	$y = \pm \frac{b}{a}x \rightarrow y = \pm \frac{4}{3}x \rightarrow$ $y = \frac{4}{3}x, \quad y = -\frac{4}{3}x$
Directrices	$y = \pm \frac{b}{e} \rightarrow y = \pm \frac{16}{5} = \pm 3.2$

Example(3.16): If the vertices of hyperbola are $(2, 0)$ and $(-2, 0)$ and the foci are $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$, find the equation of the hyperbola.

Solution:

Since the y – *coordinates* are the same and equal, then it is a horizontal hyperbola with center $c(0, 0)$ with the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and so the vertices in that case are $(\pm a, 0) = (\pm 2, 0)$ and then $a = 2$. Also, the foci are $(\pm m, 0) = (\pm \sqrt{13}, 0)$ and then $m = \sqrt{13}$.

Since $ae = m$, $m^2 = a^2 + b^2$, then $e = \frac{\sqrt{13}}{2} > 1$, $b = \sqrt{13 - 4} = \sqrt{9} = 3$.

Hence,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \rightarrow$$

$$\frac{x^2}{2^2} - \frac{y^2}{3^2} = 1 \rightarrow$$

the standard equation is:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1 .$$

Then, the general equation is:

$$9 x^2 - 4 y^2 = 36 \rightarrow 9 x^2 - 4 y^2 - 36 = 0 .$$

Homework (3.3):

1- Sketch the hyperbola with equation $4x^2 - 3y^2 - 8x + 6y - 11 = 0$.

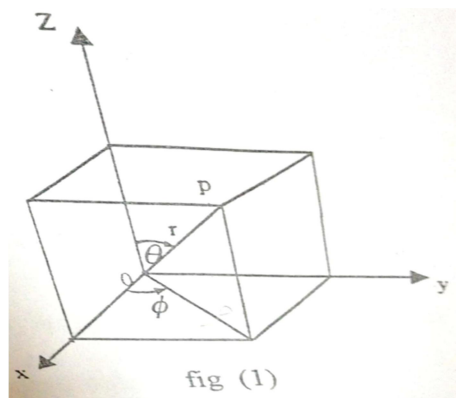
Chapter (3): Solid Geometry (3D Geometry)

In this chapter, we shall see and go through the concepts of the point, the straight lines and the sphere in 3D-space which we denote by R^3 . Some focus will be given on some important surfaces in R^3 .

A. The Cartesian, Cylindrical and Spherical Coordinates:

We represent the point in space R^3 by three coordinates.

The three planes $x=0$, $y=0$ and $z=0$ are intersecting at a point o called origin and the three lines are called coordinate plane (horizontal, vertical and side view plane).



The distance between the point P and these planes give the three coordinates of this point (x, y, z) . These coordinate planes divide the space into eight parts, for example the point $(1, 4, 2)$ lies on first octant.

The point P is determined by intersecting the plane $\phi=\text{constant}$ (it is a plane passing through the vertical of axis OZ and the measure of the angle between it and the plane XOZ is equal to this constant) and the cylinder $\rho=\text{constant}$ (it is a right circular cylinder whose axis is OZ) and the plane $Z=\text{constant}$. Hence the cylindrical coordinates are (ρ, ϕ, Z) where $0 \leq \rho < \infty$, $0 \leq \phi < 2\pi$, $-\infty < Z < \infty$.

Conversion from Cartesian to cylindrical coordinates and vice-versa.

From fig. (1), we can deduce that:

$$\begin{aligned}x &= \rho \cos \varphi & \rho &= \sqrt{x^2 + y^2} \\y &= \rho \sin \varphi & \varphi &= \tan^{-1} \left(\frac{y}{x} \right) \\Z &= Z & Z &= Z\end{aligned}$$

Example(4.1): Represent the point $P_1(3, -3, 4)$ in cylindrical coordinates and the point $P_2(5, \frac{2\pi}{3}, -7)$ in Cartesian coordinates.

Solution

For the point P_1 , $x_1 = 3$ and $y_1 = -3$ and $z_1 = 4$.

So, $\rho_1 = \sqrt{9 + 9} = \sqrt{18} = 3\sqrt{2} = 4.242$

$$\varphi_1 = \tan^{-1} \left(-\frac{3}{3} \right) = \tan^{-1}(-1) = 315^\circ \text{ or } -45^\circ$$

$Z_1 = 4$ Hence P_1 is $(4.242, 315^\circ, 4)$

For the point P_2 , $\rho_2 = 5$, $\varphi_2 = \frac{2\pi}{3}$, $Z_2 = -7$

$$x_2 = \rho_2 \cos \varphi_2 = 5 \cos \frac{2\pi}{3} = -2.5$$

$$y_2 = \rho_2 \sin \varphi_2 = 5 \sin \frac{2\pi}{3} = 4.3303$$

$$Z_2 = -7$$

Hence, P_2 is $(-2.5, 4.3303, -7)$.

(*) Spherical polar coordinates:

The point P is determining by intersecting the plane $\varphi = \text{constant}$ and the sphere $r = \text{constant}$ (it is a sphere whose center is origin and radius $r = \text{constant}$) and the right circular cone $\theta = \text{constant}$ (a cone whose vertex is origin and its axis is OZ and half vertical angle $\theta = \text{constant}$). Hence the spherical polar coordinates (r, θ, φ) where $0 \leq r < \infty$, $0 \leq \theta < \pi$, $0 < \varphi < 2\pi$.

Conversion from Cartesian to spherical polar coordinates and vice versa

From Fig. (1) we can deduce that:

$$\begin{aligned}x &= r \sin \theta \cos \varphi & r &= \sqrt{x^2 + y^2 + z^2} \\y &= r \sin \theta \sin \varphi & \theta &= \tan^{-1} \left(\sqrt{(x^2 + y^2)/z^2} \right) \\z &= r \cos \theta & \varphi &= \tan^{-1} \left(\frac{y}{x} \right)\end{aligned}$$

Example(4.2): Find the Cartesian coordinates of the point $P_1(5, 30^\circ, 120^\circ)$ and the spherical polar coordinates of the point $P_2(5, -5, 5\sqrt{2})$.

Solution:

For the point P_1 , $r_1 = 5$, $\theta_1 = 30^\circ$ and $\varphi_1 = 120^\circ$

$$x_1 = r_1 \sin \theta_1 \cos \varphi_1 = 5 \sin 30^\circ \cos 120^\circ = -1.25$$

$$y_1 = r_1 \sin \theta_1 \sin \varphi_1 = 5 \sin 30^\circ \sin 120^\circ = 2.1651$$

$$z_1 = r_1 \cos \theta_1 = 5 \cos 30^\circ = 4.3302.$$

The point P_1 is $(-1.25, 2.1651, 4.3302)$

For the point P_2 , $x_2 = 5$, $y_2 = -5$ and $z_2 = 5\sqrt{2}$.

$$\text{So } r_2 = \sqrt{25 + 25 + 50} = 10 \quad \theta_2 = \tan^{-1} \sqrt{\frac{25+25}{50}} = \tan^{-1}(1) = 45^\circ$$

$$\varphi_2 = \tan^{-1} \left(-\frac{5}{5} \right) = \tan^{-1}(-1) = 315^\circ$$

Hence the point P_2 is $(10, 45^\circ, 315^\circ)$.

Conversion from spherical to cylindrical coordinates and vice versa

From fig. (1) we can deduce that:

$$r = \sqrt{\rho^2 + z^2} \quad \rho = r \sin \theta$$

$$\theta = \tan^{-1} \left(\frac{\rho}{z} \right) \text{ and } \varphi = \varphi$$

$$\varphi = \varphi \quad z = r \cos \theta$$

Example(4.3): Find the spherical polar coordinates of the point $P_1(6, 60^\circ, -8)$ and the cylindrical coordinates of the point $P_2(5, 90^\circ, 60^\circ)$.

Solution:

From the point

$$P_1, \rho_1 = 6, \varphi_1 = 60^\circ \text{ and } Z_1 = -8$$

$$r_1 = \sqrt{36 + 64} = 10$$

$$\theta_1 = \tan^{-1} \left(\frac{6}{-8} \right) = \tan^{-1}(-0.75) = 36^\circ 52'$$

$$\varphi_1 = 60^\circ$$

Thus the point P_1 is $(10, 36^\circ 52', 60^\circ)$

For the point P_2 , $r_2 = 5$, $\theta_2 = 90^\circ$, and $\varphi_2 = 60^\circ$.

$$\rho_2 = r_2 \sin \theta_2 = 5 \sin 90^\circ = 5$$

$$\rho_2 = r_2 \cos \theta_2 = 5 \cos 90^\circ = 0$$

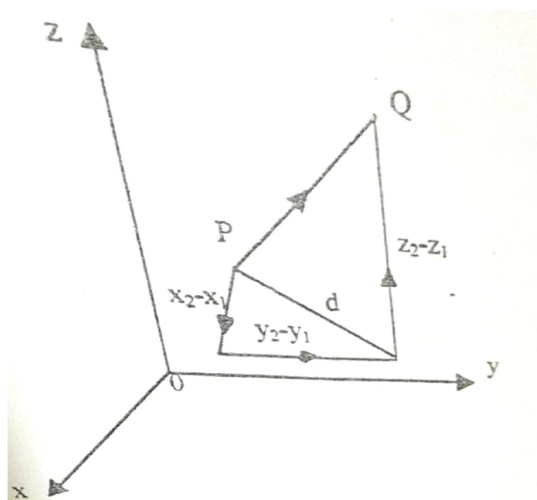
$$\varphi_2 = 60^\circ$$

The point P_2 is $(5, 60^\circ, 0)$

(*) Distance between two points

The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in terms of their coordinates

$$PQ = \sqrt{d^2 + (z_2 - z_1)^2}$$



Therefore $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \dots \dots \dots (1)$$

Special cases:

The distance between the point $P(x, y, z)$ and the origin O is

$$OP = \sqrt{x^2 + y^2 + z^2}$$

Example(4.4): Prove that the points $(-2, 4, -3)$, $(4, -3, -2)$ and $(-3, -2, 4)$ form an equilateral triangle.

Solution:

Let the points be denoted by $P(-2, 4, -3)$, $Q(4, -3, -2)$ and $R(-3, -2, 4)$

$$\ast (PQ)^2 = (4 + 2)^2 + (4 + 3)^2 + (-1)^2 = 79$$

$$(QR)^2 = (4 + 3)^2 + (-1)^2 + (-6)^2 = 79$$

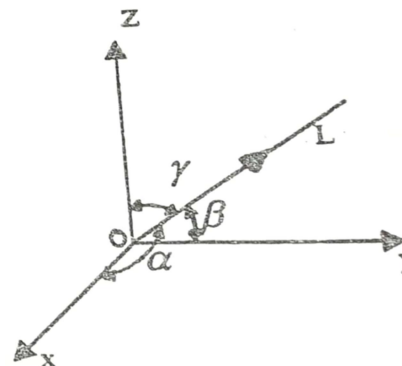
$$(PR)^2 = (1)^2 + (4 + 2)^2 + (-3 - 4)^2 = 79$$

Hence, $PQ = QR = PR \Rightarrow$ Points form an equilateral triangle.

(*) Direction Cosines of a line in the space:

If α, β, γ are direction angles of a line L , then $\cos\alpha, \cos\beta$, and $\cos\gamma$ are called direction cosine of L . In general direction cosines of the line L are denoted by l, m, n .

Thus $l = \cos\alpha, m = \cos\beta$ and $n = \cos\gamma$



And $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Note: The direction cosines of the coordinates axes, x-axis, y-axis and z-axis are respectively $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

The quantities which are proportional to the direction cosines of a line are called direction ratios (number) of the line.

(*) Relation between direction cosines and direction ratios of a line:

Let l, m, n be direction cosines of a line AB and a, b, c be its direction ratio then direction ratios of the line are proportional to its direction cosines, thus we have

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \pm \frac{1}{\sqrt{a^2+b^2+c^2}} \Rightarrow l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Note: The direction cosines of the line AB are taken with positive signs whereas for BA with negative sign in the above values of l, m, n .

Example(4.5): The direction ratios of a line are 1, 2, 3. Find its direction cosines.

Solution:

$$a = 1, b = 2, c = 3 \Rightarrow \sqrt{a^2 + b^2 + c^2} = \sqrt{14}$$

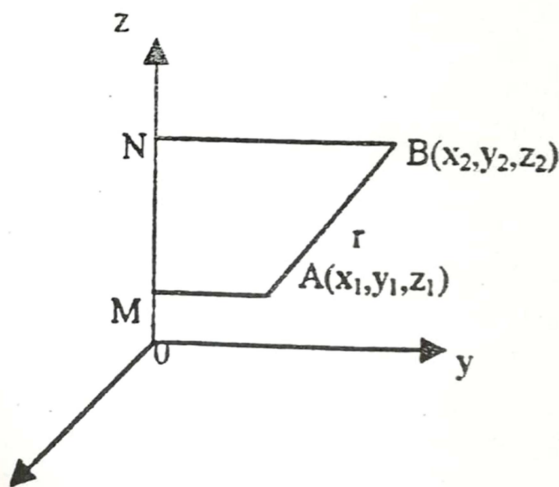
$$* \quad l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}} = \pm \frac{1}{\sqrt{14}}, \quad m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}} = \pm \frac{2}{\sqrt{14}}, \quad \text{and}$$

$$n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}} = \pm \frac{3}{\sqrt{14}}.$$

(*) Direction cosines of a line joining two points

Let l, m, n be the direction cosines of the line AB joining the points $A(x_1, y_1, z_1)$ and $A(x_2, y_2, z_2)$ and let $AB = r$.

Draw perpendiculars AM and BN on z -axis. Then $MN = \text{Projection } AB \text{ on } z$ -



$$\text{axis} = AB \cos \gamma. \quad x$$

where γ the angle between AB and z -axis but $MN = ON - OM = z_2 - z_1$, and $\cos \gamma = n$.

$$z_2 - z_1 = nr \Rightarrow n = \frac{z_2 - z_1}{r}$$

$$\text{Similarly } l = \frac{x_2 - x_1}{r}, \quad m = \frac{y_2 - y_1}{r}.$$

Hence, $\frac{x_2 - x_1}{r}, \frac{y_2 - y_1}{r}, \frac{z_2 - z_1}{r}$ are the direction cosines of the line AB .

Remark: The direction ratios of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Example(4.6): Find direction ratios (numbers) and direction cosines for the L which passing through the points $P_1(1, 5, 2)$ and $P_2(3, 7, -4)$.

Solution:

The direction ratios are: 2, 2, -6. We compute $r = \sqrt{4 + 4 + 36} = \sqrt{44} = 2\sqrt{11}$ and so

$\frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}$ form a set of direction cosines. Since L is undirected it has two such sets, the other being $-\frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}$.

(*) Angle between two lines

Let L_1 and L_2 be the two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively. The angle θ between the two lines is given by

$$\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

The two lines are parallel if: $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ and they are perpendicular if:

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0.$$

Note: A line L_1 is parallel to a line L_2 if and only if direction ratios of $l_1(a_1, b_1, c_1)$ is proportional to a set of direction numbers of $l_2(a_2, b_2, c_2)$

i.e. $\frac{a_1}{a_2} = k, \frac{b_1}{b_2} = k, \frac{c_1}{c_2} = k$ or more simply $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ where the number K may be positive or negative but not zero.

Example(4.7): Do the three points $P_1(3, -1, 4)$ and $P_2(1, 6, 8)$ and $P_3(9, -22, -8)$ on the same straight line?

Solution:

A set of direction numbers for the line L_1 through P_1 and P_2 is $(-2, 7, 4)$.

A set of direction numbers for the line L_2 through P_2 and P_3 is $(8, -28, -16)$.

Since the second set is proportional to the first set (with $K = -4$). We conclude that L_1 and L_2 have the same direction cosines. Therefore the two lines are parallel. However, they have the points P_2 in common and so must coincide.

Remark

- i. The angle θ between the lines whose direction ratios a_1, b_1, c_1 and

$$a_2, b_2, c_2 \text{ is given by } \theta = \cos^{-1} \left(\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right)$$

- ii. These lines are perpendicular if and only if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.

Example(4.8): Find the cosines of the angle between the line L_1 passing through the points $P_1(1, 4, 2)$ and $P_2(3, -1, 3)$ and the line L_2 passing through the points $Q_1(3, 1, 2)$ and $Q_2(2, 1, 3)$.

Solution:

A set of direction numbers for L_1 is $(2, -5, 1)$ and a set of direction numbers for L_2 is $(-1, 0, 1)$.

Therefore direction cosines for the two lines are

$$L_1: \frac{2}{\sqrt{30}}, \frac{-5}{\sqrt{30}}, \frac{1}{\sqrt{30}} \text{ and}$$

$$L_2: \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \text{ then we obtain}$$

$$\cos \theta = \frac{-1}{\sqrt{15}} + 0 + \frac{1}{2\sqrt{14}} = \frac{-1}{2\sqrt{15}}$$

B. The plane in the space:

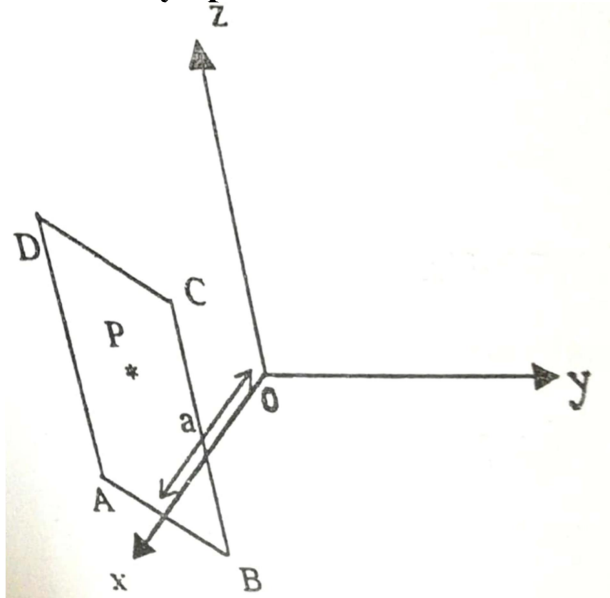
Definition: A plain is a surface such that if any two points be taken on it, the line joining them lies wholly in the surface.

3.2.1 Particular Planes

Before we come to general discussion of the equation of plane, we examine some particular cases.

The equation of a plane parallel to one of the coordinate planes are:

$X=a$ Parallel to yz -plane



$Y=b$ Parallel to zx -plane

$Z=c$ Parallel to xy -plane

Remark: The equations of the coordinate planes are

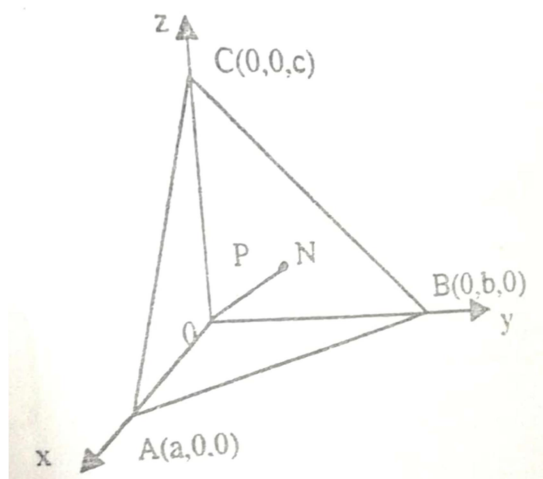
- i. **$X=0$ for yz -plane**
- ii. **$Y=0$ for zx -plane**
- iii. **$Z=0$ for xy -plane**

(*) Various forms of the equation of a plane:

A. Normal Form

Let ABC be a plane and ON the normal to the plane and equate to P . Let l, m, n be the actual direction cosines of ON and let the point (x, y, z) be any point

on the plane.



Hence the equation of a plane in terms of the length of the normal drawn from the origin to the plane and its direction cosine is

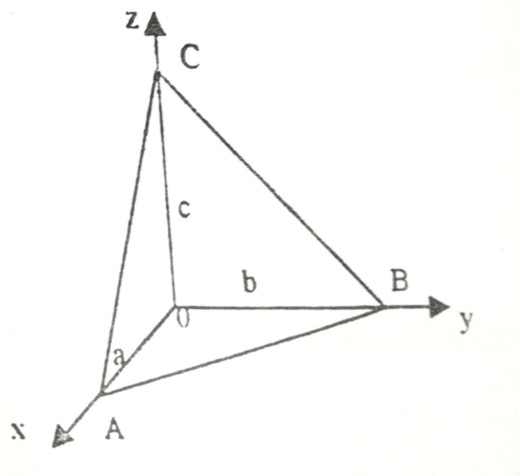
$$lx + my + nz = 0 \dots\dots\dots(1)$$

Remarks

- i. Normal form is also referred to perpendicular form.
- ii. The foot of the perpendicular drawn from the origin to the plane $lx + my + nz = p$ is (lp, mp, np) .

B. Intercept Form

Let ABC be a plane which cuts intercepts a , b , and c on the axes. Then the equation of the plane in terms of the intercepts which it makes on the axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \dots\dots\dots(2)$



(*) General Form

A plane may be represented by the general first degree in x, y equation

$$Ax + By + Cz + D = 0 \dots\dots\dots (3)$$

where the coefficients A, B, and C are real and at least one of them is not zero.

Remarks: We have the following observations about the general equation of the plane.

- i. When $A=0$, the plane is parallel to x-axis.
- ii. When $B=0$, the plane is parallel to y-axis.
- iii. When $C=0$, the plane is parallel to z-axis.
- iv. When $D=0$, the plane passes through origin.
- v. When $A=B=0$, the plane is parallel to xy-axis.
- vi. When $B=C=0$, the plane is parallel to yz-axis.
- vii. When $A=C=0$, the plane is parallel to zx-axis.
- viii. When $A=B=C=0$, D is finite the, the plane is at infinite distance.

(*) The equation of a plane passes through a given point

Let (x_1, y_1, z_1) be any given point. Let the equation of a plane be

$$Ax + By + Cz = 0.$$

Then equation of the in this case is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \dots\dots\dots (4)$$

The equation (4) is called one-point form of the plane.

In Special Cases: The equation of the plane passing through origin is:

$$Ax + By + Cz = 0$$

(*) The equation of a plane passes through 3-points

Let the points (x_1, y_1, z_1) , (x_2, y_2, z_2) and (x_3, y_3, z_3) be three points.

Let the equation of the plane be $Ax + By + Cz + D = 0$ then the plane which passes through the given three points is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \dots\dots\dots (5)$$

The equation (5) of the plane is called three –point form. Since a plane is completely specified if it passes through three given points, a fourth point may or may not lie on it. However we have the following:

- i. A necessary and sufficient condition that the four points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ and $P_4(x_4, y_4, z_4)$, no three of which are collinear shall lie in a plane is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

(*) Angle between two planes

The angle between two planes is equal to the angle which the positive direction of a normal to one plane makes with the positive direction of a normal to the other plane.

Let the equations of the given planes be $A_1x + B_1y + C_1z = D_1$(i)
and $A_2x + B_2y + C_2z = D_2$(ii)

The direction cosines of the normal to the given planes are respectively proportional to A_1, B_1, C_1 and A_2, B_2, C_2 .

Let θ be the angle between the given planes. Then

$$\cos \theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}}$$

Notes: A plane is said to be parallel to a line if any perpendicular to the line is a normal to the plane.

A plane is said to be parallel to another plane if normal of one plane are also the normal of the other.

Hence, the plane (i) and (ii) are parallel if and only if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \text{ and are perpendicular if and only if } A_1A_2 + B_1B_2 + C_1C_2 = 0$$

Example(4.9): Find the equation of the plane which passes through the point $(2, -3, 4)$ and parallel to the plane $2x - 5y - 7z = 0$.

Solution:

Any plane parallel to the given plane is $2x - 5y - 7z + \lambda = 0$. This will pass through the point $(2, -3, 4)$ if

$$2(2) - 5(-3) - 7(4) + \lambda = 0 \Rightarrow \lambda = 9.$$

Therefore the required equation of the plane is $2x - 5y - 7z + 9 = 0$.

Example(4.10): Find the equation of the plane through the point $(-1, 3, 2)$ and perpendicular to the planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$.

Solution:

Let the equation of any plane is

$$Ax + By + Cz + D = 0 \dots\dots\dots(i)$$

This plane will be perpendicular to the given plane if

$$A + 2B + 2C = 0 \dots\dots\dots(ii)$$

$$\text{and } 3A + 3B + 2C = 0 \dots\dots\dots(iii)$$

Since the plane (i) will be passing through $(-1, 3, 2)$

$$\text{Therefore } -A + 3B + 2C + D = 0 \dots\dots\dots(iv)$$

From equation (ii) and (iii) we get

$$\frac{A}{2} = \frac{B}{-4} = \frac{C}{3} \dots\dots\dots(v)$$

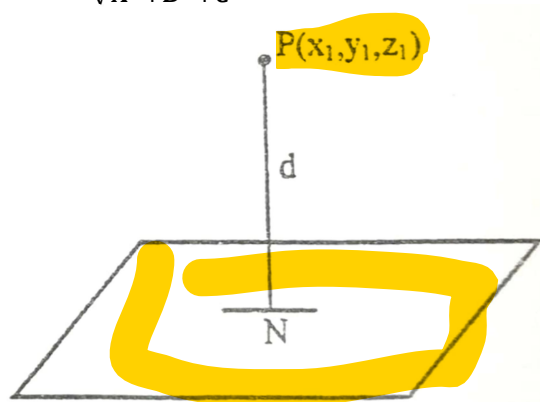
From (iv) and (v) we get $D = -8$. Hence the required equation of the plane becomes

$$2x - 4y + 3z + 8 = 0$$

(*) **Perpendicular distance** of a point from a plane

From the point $P(x_1, y_1, z_1)$ draw the normal PN to the given plane $Ax + By + Cz + D = 0$.

$$\text{Then } PN = \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}$$



Let the length of the normal is d : then $d = \left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \right|$

In special cases: The perpendicular distance (shortest distance) of the origin from the plane $Ax + By + Cz + D = 0$ is $d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$.

Example(4.11): Find the length of the normal from the point (2, -3, 6) to the plane $4x + 3y - 5z = 7$.

Solution:

The required distance is

$$d = \left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \right| = \left| \frac{2(4) + 3(-3) + 5(6) - 7}{\sqrt{16 + 9 + 25}} \right| = 5.373.$$

Example(4.12): Find the distance of the origin from the plane $6x - 3y + 2z - 14 = 0$.

Solution: The required distance is

$$d = \left| \frac{D}{\sqrt{A^2 + B^2 + C^2}} \right| = \left| \frac{-14}{\sqrt{36 + 9 + 4}} \right| = \frac{14}{7} = 2.$$

(*) Bisectors of angles:

Bisector of angles between the two planes is the locus of a point which moves such that its distance from two planes are equal in magnitude.

Let $A_1x + B_1y + C_1z + D_1 = 0$ and

$A_2x + B_2y + C_2z + D_2 = 0$ be any two given intersecting planes. Then the equations of the planes which bisect the angles between the given planes are:

$$\frac{A_1x + B_1y + C_1z + D_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \pm \frac{A_2x + B_2y + C_2z + D_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}} \dots\dots\dots(i)$$

The equation (i) with positive sign in the right hand side is called internal bisector and with negative sign is called external bisector.

Remark:

The planes representing the internal and external bisectors of the angles between any two intersecting planes, of course intersecting are perpendicular to each other.

Example(4.13): Find the equations of the planes bisecting the angles between the planes

$$3x + 4y - 5z + 1 = 0 \text{ and } 5x + 12y - 13z = 0.$$

Solution:

The equation of the internal bisector is

$$\frac{3x+4y-5z+1}{\sqrt{3^2+4^2+(-5)^2}} = \frac{5x+12y-13z}{\sqrt{5^2+12^2+(-13)^2}}$$

$$\Rightarrow 14x-8y+13=0 \dots\dots\dots(a)$$

Further, the equation of the other bisector (external) is

$$\frac{3x+4y-5z+1}{\sqrt{3^2+4^2+(-5)^2}} = -\frac{5x+12y-13z}{\sqrt{5^2+12^2+(-13)^2}}$$

$$\Rightarrow 64x-112y-130z+13=0 \dots\dots\dots(b)$$

The two planes (a) and (b) are perpendicular because

$$A_1A_2 + B_1B_2 + C_1C_2 = 0. \text{ i.e. } 14(64) - 8(112) - (130)0 = 0$$

(*) System of Planes:

The collection of all planes defined by a linear equation with one or two parameter is called a family or system of planes.

A system is said to be one-parameter or two parameters system according as its equation has, respectively one or two parameters as follows:

i. System of planes parallel to a given plane

Let the equation of the given plane be $Ax+By+Cz+D=0$. Then the system of planes parallel to the given plane is

$Ax+By+Cz+\lambda=0$ where λ is a parameter. This is one-parameter system of planes.

ii. System of planes through the intersection of two planes.

Let the two distinct intersecting planes be

$$A_1x + B_1y + C_1z + D_1 = 0 \dots\dots\dots(i)$$

$$\text{And } A_2x + B_2y + C_2z + D_2 = 0 \dots\dots\dots(ii)$$

Then the equation

$(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2) = 0$ where λ is a parameter, represents a system of planes, through the line of intersection of planes (i) and (ii). This is also one-parameter system.

iii. System of planes through a given point

Let the given point is (x_1, y_1, z_1) . Then the equation of any plane passing through this point is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

If $A \neq 0$, dividing by A throughout, we get

$$(x - x_1) + \frac{B}{A}(y - y_1) + \frac{C}{A}(z - z_1) = 0$$

Putting $\lambda = \frac{B}{A}$ and $\mu = \frac{C}{A}$, we obtain

$$(x - x_1) + \lambda(y - y_1) + \mu(z - z_1) = 0$$

This is the equation of the system of planes each member of which passes through (x_1, y_1, z_1) where λ and μ are two parameters.

This is a two-parameters system of planes.

Example(4.14): Find the equation of the plane passing through the line of intersection of the planes

$x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$.

Solution:

The equation of the plane passing through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ is $(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0$

i.e. $(1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - 6 + 5\lambda = 0 \dots \dots \dots (i)$

This plane will be perpendicular to the plane $4x + 5y - 3z = 8$ if

$4(1 + 2\lambda) + 5(1 + 3\lambda) - 3(1 + 4\lambda) = 0$ which gives $\lambda = -\frac{6}{11}$. Substituting

this value of λ in (i) we get $x + 7y + 13z + 96 = 0$.

This is required equation of the plane.

C. The Straight Lines in the space:

The intersection of two surfaces S_1 and S_2 is the locus of all the common points on both the surfaces S_1 and S_2 . This locus, in general would be a surface which in particular case could be a plane, a plane curve, a line, a single point or even no point. In most of the cases of which is common use in analytical geometry the intersection of two surfaces S_1 and S_2 is a space curve. The curve of intersection of a plane and a surface is called a plane section of the surface. Furthermore, if both the surfaces are taken as planes, their intersection, in general, would be a line. Conversely, any given line can be uniquely determined by any of the two planes containing the line.

Thus a line can be regarded as the locus of the common points of two intersecting planes.

Now consider the equations of the two planes

$$A_1x + B_1y + C_1z + D_1 = 0 \dots\dots\dots(i)$$

And $A_2x + B_2y + C_2z + D_2 = 0 \dots\dots\dots(ii)$

These two equations taken together would represent the equation of a line. The form of the equation given above is called a non-symmetrical form of the equation of a line. This form is called general equation of a line.

Note: There may be a case when the two planes are non-intersecting in that case the line represented by the above equations would be an imaginary one called the virtual line of intersection of the two planes.

(*) Various forms of the equations of a line:

1. One-Point Form

The equations of the line passing through a point $A(\alpha, \beta, \gamma)$ and having direction cosines l, m, n . Let $P(x, y, z)$ be any point on the line such that $AP = r$. Then

$$\frac{x - \alpha}{r} = l, \quad \frac{y - \beta}{r} = m, \quad \frac{z - \gamma}{r} = n$$

These equations may also be written as

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} (= r) \dots\dots\dots(1)$$

This is also called the **symmetrical form** of equations of a line.

Notes:

- i. If l, m, n are taken as direction ratios instead of the direction cosines, the form of the equations of a line remain the same.
- ii. The form of the equation (1) of a line suggests use to express the coordinates of any point on the line in terms of a single parameter as

$$x = \alpha + lr, \quad y = \beta + mr, \quad z = \gamma + nr.$$

Example(4.15): Find the equations of a line passing through $(1, 2, 3)$ and having direction cosines $2, 4, 6$.

Solution:

The equations of the line are

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \text{ where } (\alpha, \beta, \gamma) = (1, 2, 3) \text{ and } l = 2, m = 4, n = 6.$$

Therefore

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{6}.$$

2. Two Point Form

The equations of a line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \text{ or } \frac{x-x_2}{x_2-x_1} = \frac{y-y_2}{y_2-y_1} = \frac{z-z_2}{z_2-z_1} \dots\dots\dots(2)$$

Example(4.16):

Find the equations of the line joining the points $(1, 2, 3)$ and $(1, 4, 6)$

Solution:

The required equations of the line are

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3} \text{ or } \frac{x-2}{1} = \frac{y-4}{2} = \frac{z-6}{3} \text{ from (2)}$$

3. Reduction of general form to symmetrical form

To transfer the equations

$$A_1x + B_1y + C_1z + D_1 = 0 \text{ and } A_2x + B_2y + C_2z + D_2 = 0 \dots\dots(i)$$

of a line into symmetrical form.

The line parallel to (i) passing through origin is given by

$$A_1x + B_1y + C_1z = 0 \text{ and } A_2x + B_2y + C_2z = 0 \dots\dots\dots(ii)$$

The equation (ii) may be written as

$$\frac{x}{B_1C_2 - B_2C_1} = \frac{y}{C_1A_2 - C_2A_1} = \frac{z}{A_1B_2 - A_2B_1}$$

Thus the direction cosines of the line (ii) and hence of (i) are proportional to

$B_1C_2 - B_2C_1$, $C_1A_2 - C_2A_1$, $A_1B_2 - A_2B_1$. Now we need one point on the line (i). In fact, there are infinite numbers of points on the line. For simplicity we choose the point of intersection of the line (i) with the plane $Z = 0$ whose coordinates are given by the equations

$$A_1x + B_1y + C_1z = 0$$

$$A_2x + B_2y + C_2z = 0$$

$$z = 0$$

By solving these equations we get the point

$$\left[\frac{B_1D_2 - B_2D_1}{A_1B_2 - A_2B_1}, \frac{D_1A_2 - D_2A_1}{A_1B_2 - A_2B_1}, 0 \right].$$

Thus the equations of the line in symmetrical form are

$$\frac{x - \frac{B_1D_2 - B_2D_1}{A_1B_2 - A_2B_1}}{B_1C_2 - B_2C_1} = \frac{y - \frac{D_1A_2 - D_2A_1}{A_1B_2 - A_2B_1}}{C_1A_2 - C_2A_1} = \frac{z}{A_1B_2 - A_2B_1}.$$

Note: It is not necessary to take $z = 0$ always for finding a point on the line. This depends upon the convenience. In some cases there may be need to take either $x = 0$ or $y = 0$. It is also possible to take some suitable value of x, y and z different from zero.

Example(4.17): Transform the equations

$4x + 4y - 5z - 12 = 0$ and $8x + 12y - 13z - 32 = 0$(i)
of a line into symmetrical form.

Solution:

The line parallel to (i) and passing through the origin is given by

$$4x + 4y - 5z - 12 = 0 \quad \text{and} \quad 8x + 12y - 13z - 32 = 0$$

.....(ii)

These equations may be written as

$$\frac{x}{-52 + 60} = \frac{y}{-40 + 52} = \frac{z}{48 - 32}$$

i.e. $\frac{x}{8} = \frac{y}{12} = \frac{z}{16}$

i.e. $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$.

Thus the direction cosines of the line (ii) and hence of (i) proportional to 2, 3, 4. Now, we find the **point of intersection** of the line (i) with the plane $Z = 0$ whose coordinates are given by

$$4x + 4y - 5z - 12 = 0$$

$$8x + 12y - 13z - 32 = 0$$

$$Z = 0.$$

By solving these equations we get $x = 1, y = 2, z = 0$. Thus the equation of the line in symmetrical form are

$$\frac{x - 1}{2} = \frac{y - 2}{3} = \frac{z}{4}$$

Example(4.18): For where the line:

$$\frac{x - 1}{2} = \frac{y - 2}{-3} = \frac{z + 3}{4}$$

meets the plane $2x + 4y - z + 1 = 0$.

Solution: Let $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4} = r$.

Then any point on the line $[1 + 2r, 2 - 3r, -3 + 4r]$ if this point lies on the plane $2x + 4y - z + 1 = 0$ then $2x + 4y - z + 1 = 0$.

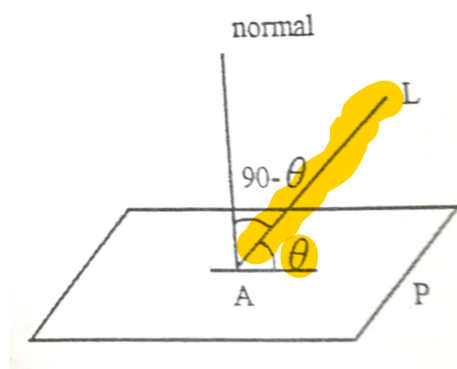
Then $2(1 + 2r) + 4(2 - 3r) - (-3 + 4r) + 1 = 0 \Rightarrow r = \frac{7}{6}$.

Thus the point of intersection of the line with the plane is $(\frac{10}{3}, -\frac{3}{2}, \frac{5}{3})$.

(*) Plane and a line:

1. Angle between a plane and a line

Suppose a line L meets the plane P at the point A . Then the acute angle between the line L and its projection on the plane P is called the angle between the line L and the plane P .



Let the equation of the given plane and the line be

$$Ax + By + Cz + D = 0 \dots\dots\dots (1)$$

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \dots\dots\dots (2)$$

Let θ be the angle between the plane (1) and the line (2). Then the angle between the normal to the plane (1) and the line (2) is $(90^\circ - \theta)$. Also the direction cosines of the normal to the plane are proportional to A, B, C whereas those of the line are l, m, n .
Thus,

$$\sin \theta = \frac{Al + Bm + Cn}{\sqrt{A^2 + B^2 + C^2} \sqrt{l^2 + m^2 + n^2}}$$

Notes:

- i. The line is parallel to the plane if :

$$Al + Bm + Cn = 0$$

Which is consistent with the condition that the normal to the plane is perpendicular to the line.

- ii. The line is perpendicular to the plane if:

$$\frac{A}{l} = \frac{B}{m} = \frac{C}{n}$$

Which is consistent with the condition that the normal to the plane is parallel to the line.

2. Conditions for a line to lie in a plane

Let the equations of the line and the plane be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} (= r) \dots\dots\dots(1)$$

$$\text{and } Ax + By + Cz + D = 0 \dots\dots\dots(2)$$

Then, the conditions that the line (1) may be lie in the plane (2) are

$$Al + Bm + Cn = 0 \text{ and } A\alpha + B\beta + C\gamma + D = 0$$

Remarks:

- i. The condition $Al + Bm + Cn = 0$ shows that the line is perpendicular to the normal to the plane and hence parallel to the plane.

- ii. The condition $A\alpha + B\beta + C\gamma + D = 0$ shows that the point (α, β, γ) lies on the plane.

Thus for a line lie in a plane it is necessary and sufficient that the point (α, β, γ) lies on the plane.

Thus for a line lie in a plane it is necessary and sufficient that the line should be parallel to the line and one point of the line must lie on the plane.

3. Plane Containing a line

The equation of the plane which contains the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \text{ is}$$

$$A(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0 \text{ where } Al + Bm + Cn = 0.$$

Example 3.19

Example(4.19): Prove that the line $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ lies on the plane $4x + 4y - 5z - 3 = 0$.

Solution: From the given equations of the line and the plane:

$$\begin{aligned} A &= 4, & B &= 4, & C &= -5, & D &= -3 \text{ and} \\ \alpha &= 2, & \beta &= 4, & \gamma &= 5 \\ l &= 2, & m &= 3, & n &= 4. \end{aligned}$$

The conditions that the lines lie on the plane are

$$Al + Bm + Cn = 0$$

$$\Rightarrow 8 + 12 - 20 = 0, \text{ and}$$

$$A\alpha + B\beta + C\gamma + D = 0 \Rightarrow 12 + 16 - 25 - 3 = 0. \text{ Hence the conditions are satisfied.}$$

Example(4.20): Find the equation of the plane containing the line

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4} \text{ and perpendicular to the plane } x + 2y + z = 12.$$

Solution:

The equation of the plane is $A(x - \alpha) + B(y - \beta) + C(z - \gamma) = 0$.

Here $A = 1$, $B = 2$, $C = 1$ and $\alpha = 1$, $\beta = -1$, $\gamma = 3$

So equation of the plane is $1(x - 1) + 2(y + 1) + 1(z - 3) = 0$

$$\Rightarrow x + 2y + z = 2.$$

References:

- 1- A first course in Differential Geometry, International Press, Cambridge, MA, (1997).