

Chapter 1

Definite Integral

2.1 Summation Notation

Definition 2.1 Let $\{a_1, a_2, \dots, a_n\}$ be a set of numbers. The symbol $\sum_{k=1}^n a_k$ represents their sum:

$$\sum_{k=1}^n a_k = a_1 + a_2 + \dots + a_n.$$

Example 2.1 Evaluate the sum.

(1) $\sum_{i=1}^3 i^3$

(2) $\sum_{j=1}^4 (j^2 + 1)$

Solution:

(1) $\sum_{i=1}^3 i^3 = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36.$

(2) $\sum_{j=1}^4 (j^2 + 1) = (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) = 2 + 5 + 10 + 17 = 34.$

Theorem 2.1 Let $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ be sets of real numbers. If n is any positive integer, then

1. $\sum_{k=1}^n c = \underbrace{c + c + \dots + c}_{n\text{-times}} = nc$ for any $c \in \mathbb{R}$.

2. $\sum_{k=1}^n (a_k \pm b_k) = \sum_{k=1}^n a_k \pm \sum_{k=1}^n b_k.$

3. $\sum_{k=1}^n c a_k = c \sum_{k=1}^n a_k$ for any $c \in \mathbb{R}$.

Definition 2.2 A set $P = \{x_0, x_1, x_2, \dots, x_n\}$ is called a partition of a closed interval $[a, b]$ if for any positive integer n ,

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

Definition 2.3 The norm of the partition of P is the largest length among $\Delta x_1, \Delta x_2, \Delta x_3, \dots, \Delta x_n$ i.e.,

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \Delta x_3, \dots, \Delta x_n\}.$$



Remark 2.1

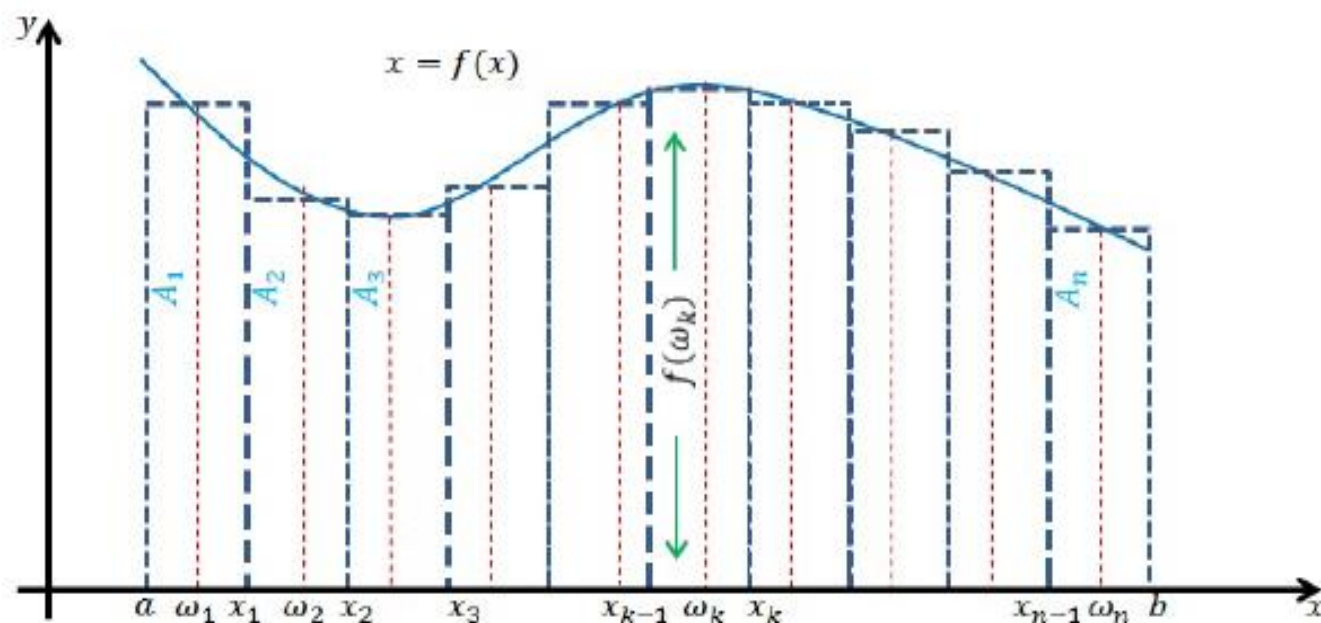
1. The partition P of the interval $[a, b]$ is regular if $\Delta x_0 = \Delta x_1 = \Delta x_2 = \dots = \Delta x_n = \Delta x$.
2. For any positive integer n , if the partition P is regular then

$$\Delta x = \frac{b-a}{n} \text{ and } x_k = x_0 + k \Delta x.$$

Now, we are ready to define a Riemann sum .

Definition 2.4 Let f be a function defined on a closed interval $[a, b]$ and let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Let $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is a mark on the partition P where $\omega_k \in [x_{k-1}, x_k]$, $k = 1, 2, 3, \dots, n$. Then, a Riemann sum of f for P is

$$R_p = \sum_{k=1}^n f(\omega_k) \Delta x_k.$$



Example 2.7 Find a Riemann sum R_p of the function $f(x) = 2x - 1$ for the partition $P = \{-2, 0, 1, 4, 6\}$ of the interval $[-2, 6]$ by choosing the mark,

- (1) the left-hand endpoint,
- (2) the right-hand endpoint,
- (3) the midpoint.

Solution:

- (1) Choose the left-hand endpoint of each subinterval.

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k) \Delta x_k$
$[-2, 0]$	$0 - (-2) = 2$	-2	-5	-10
$[0, 1]$	$1 - 0 = 1$	0	-1	-1
$[1, 4]$	$4 - 1 = 3$	1	1	3
$[4, 6]$	$6 - 4 = 2$	4	7	14
$R_p = \sum_{k=1}^4 f(\omega_k) \Delta x_k$				6

- (2) Choose the right-hand endpoint of each subinterval.

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k) \Delta x_k$
$[-2, 0]$	$0 - (-2) = 2$	0	-1	-2
$[0, 1]$	$1 - 0 = 1$	1	1	1
$[1, 4]$	$4 - 1 = 3$	4	7	21
$[4, 6]$	$6 - 4 = 2$	6	11	22
$R_p = \sum_{k=1}^4 f(\omega_k) \Delta x_k$				42

- (3) Choose the midpoint of each subinterval.¹

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k) \Delta x_k$
$[-2, 0]$	$0 - (-2) = 2$	-1	-3	-6
$[0, 1]$	$1 - 0 = 1$	0.5	0	0
$[1, 4]$	$4 - 1 = 3$	2.5	4	12
$[4, 6]$	$6 - 4 = 2$	5	9	18
$R_p = \sum_{k=1}^4 f(\omega_k) \Delta x_k$				24

$$^1 \omega_k = \frac{x_{k-1} + x_k}{2}.$$

Definition 2.5 Let f be a defined function on a closed interval $[a, b]$ and let P be a partition of $[a, b]$. The definite integral of f on $[a, b]$ is

$$\int_a^b f(x) \, dx = \lim_{\|P\| \rightarrow 0} \sum_k f(\omega_k) \Delta x_k$$

if the limit exists. The numbers a and b are called the limits of the integration.

Example 2.9 Evaluate the integral $\int_2^4 (x+2) \, dx$.

Solution:

Let $P = \{x_0, x_1, \dots, x_n\}$ be a regular partition of the interval $[2, 4]$, then $\Delta x = \frac{4-2}{n} = \frac{2}{n}$ and $x_k = x_0 + \Delta x$. Also, let the mark ω be the right endpoint of each subinterval, so $\omega_k = x_k = 2 + \frac{2k}{n}$ and then $f(\omega_k) = \frac{2}{n}(2n+k)$.

The Riemann sum of f for P is

$$R_p = \sum_k f(\omega_k) \Delta x_k = \frac{4}{n^2} \sum_k (2n+k) = \frac{4}{n^2} \left(2n^2 + \frac{n(n+1)}{2} \right) = 8 + \frac{2(n+1)}{n}.$$

From Definition 2.5, $\int_2^4 (x+2) \, dx = \lim_{n \rightarrow \infty} R_p = 8 + \lim_{n \rightarrow \infty} \frac{2(n+1)}{n} = 8 + 2 = 10$.

Properties of the Definite Integral

Theorem 2.3

1. $\int_a^b c \, dx = c(b-a),$
2. $\int_a^a f(x) \, dx = 0$ if $f(a)$ exists.

1. If f and g are integrable on $[a, b]$, then $f + g$ and $f - g$ are integrable on $[a, b]$ and

$$\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx.$$

2. If f is integrable on $[a, b]$ and $k \in \mathbb{R}$, then $k f$ is integrable on $[a, b]$ and

$$\int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx.$$

1. If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx.$$

2. If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$, then

$$\int_a^b f(x) \, dx \geq 0.$$

Theorem 2.6 If f is integrable on the intervals $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$

If f is integrable on $[a, b]$, then

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx.$$

Example 2.10 Evaluate the integral.

(1) $\int_0^2 3 \, dx$

(2) $\int_2^2 (x^2 + 4) \, dx$

Solution:

(1) $\int_0^2 3 \, dx = 3(2 - 0) = 6.$

(2) $\int_2^2 (x^2 + 4) \, dx = 0.$

Example 2.11 If $\int_a^b f(x) dx = 4$ and $\int_a^b g(x) dx = 2$, then find $\int_a^b \left(3f(x) - \frac{g(x)}{2}\right) dx$.

Solution:

$$\int_a^b \left(3f(x) - \frac{g(x)}{2}\right) dx = 3 \int_a^b f(x) dx - \frac{1}{2} \int_a^b g(x) dx = 3(4) - \frac{1}{2}(2) = 11.$$

II) The Fundamental Theorem of Calculus

Definition 1.1 A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x) \text{ for every } x \in I.$$

Theorem 2.8 Suppose that f is continuous on the closed interval $[a, b]$.

1. If $F(x) = \int_a^x f(t) dt$ for every $x \in [a, b]$, then $F(x)$ is an antiderivative of f on $[a, b]$.
2. If $F(x)$ is any antiderivative of f on $[a, b]$, then $\int_a^b f(x) dx = F(b) - F(a)$.

Corollary 2.1 If F is an antiderivative of f , then

$$\int_a^b f(x) dx = \left[F(x)\right]_a^b = F(b) - F(a).$$

Example 2.13 Evaluate the integral.

(1) $\int_{-1}^2 (2x+1) dx$

(4) $\int_0^{\frac{\pi}{2}} (\sin x + 1) dx$

(2) $\int_0^3 (x^2 + 1) dx$

(5) $\int_{\frac{\pi}{4}}^{\pi} (\sec^2 x - 4) dx$

(3) $\int_1^2 \frac{1}{\sqrt{x^3}} dx$

(6) $\int_0^{\frac{\pi}{3}} (\sec x \tan x + x) dx$

Solution:

$$(1) \int_{-1}^2 (2x+1) dx = \left[x^2 + x \right]_{-1}^2 = (4+2) - ((-1)^2 + (-1)) = 6 - 0 = 6.$$

$$(2) \int_0^3 (x^2 + 1) dx = \left[\frac{x^3}{3} + x \right]_0^3 = \left(\frac{27}{3} + 3 \right) - 0 = 12.$$

$$(3) \int_1^2 \frac{1}{\sqrt{x^3}} dx = \left[\frac{-2}{\sqrt{x}} \right]_1^2 = \frac{-2}{\sqrt{2}} - (-2) = \frac{-2+2\sqrt{2}}{\sqrt{2}} = -\sqrt{2} + 2.$$

$$(4) \int_0^{\frac{\pi}{2}} (\sin x + 1) dx = \left[-\cos x + x \right]_0^{\frac{\pi}{2}} = \left(-\cos \frac{\pi}{2} + \frac{\pi}{2} \right) - (-\cos 0 + 0) = \frac{\pi}{2} + 1.$$

$$(5) \int_{\frac{\pi}{4}}^{\pi} (\sec^2 x - 4) dx = \left[\tan x - 4x \right]_{\frac{\pi}{4}}^{\pi} = (\tan \pi - 4\pi) - \left(\tan \frac{\pi}{4} - 4\frac{\pi}{4} \right) = -4\pi - (1 - \pi) = -3\pi - 1.$$

$$(6) \int_0^{\frac{\pi}{3}} (\sec x \tan x + x) dx = \left[\sec x + \frac{x^2}{2} \right]_0^{\frac{\pi}{3}} = \left(\sec \frac{\pi}{3} + \frac{(\frac{\pi}{3})^2}{2} \right) - \left(\sec 0 + \frac{0}{2} \right) = 2 + \frac{\pi^2}{18} - 1 = 1 + \frac{\pi^2}{18}.$$

Example 2.14 If $f(x) = \begin{cases} x^2 & : x < 0 \\ x^3 & : x \geq 0 \end{cases}$, find $\int_{-1}^2 f(x) dx$.

Solution:

The definition of the function f changes at 0. Since $[-1, 2] = [-1, 0] \cup [0, 2]$, then from Theorem 2.6,

$$\begin{aligned} \int_{-1}^2 f(x) dx &= \int_{-1}^0 f(x) dx + \int_0^2 f(x) dx \\ &= \int_{-1}^0 x^2 dx + \int_0^2 x^3 dx \\ &= \left[\frac{x^3}{3} \right]_{-1}^0 + \left[\frac{x^4}{4} \right]_0^2 \\ &= \frac{1}{3} + \frac{16}{4} = \frac{13}{3}. \end{aligned}$$

Example 2.15 Evaluate the integral $\int_0^2 |x - 1| \, dx$.

Solution:

$$|x - 1| = \begin{cases} -(x - 1) & : x < 1 \\ x - 1 & : x \geq 1 \end{cases}$$

Since $[0, 2] = [0, 1] \cup [1, 2]$, then from Theorem 2.6,

$$\begin{aligned} \int_0^2 |x - 1| \, dx &= \int_0^1 (-x + 1) \, dx + \int_1^2 (x - 1) \, dx \\ &= \left[\frac{-x^2}{2} + x \right]_0^1 + \left[\frac{x^2}{2} - x \right]_1^2 \\ &= \left(\frac{1}{2} - 0 \right) + \left(0 + \frac{1}{2} \right) = 1. \end{aligned}$$

Theorem 2.9 If f is continuous on a closed interval $[a, b]$, then there is at least a number $z \in (a, b)$ such that

$$\int_a^b f(x) \, dx = f(z)(b - a).$$

Example 2.16 Find a number z that satisfies the conclusion of the Mean Value Theorem for the function f on the given interval.

(I) $f(x) = 1 + x^2, \quad [0, 2]$

Solution:

(I) From Theorem 2.9,

$$\begin{aligned} \int_0^2 (1 + x^2) \, dx &= (2 - 0)f(z) \\ \left[x + \frac{x^3}{3} \right]_0^2 &= 2(1 + z^2) \\ \frac{14}{3} &= 2(1 + z^2) \\ \frac{7}{3} &= 1 + z^2 \end{aligned}$$

This implies $z^2 = \frac{4}{3}$, then $z = \pm \frac{2}{\sqrt{3}}$. However, $-\frac{2}{\sqrt{3}} \notin (0, 2)$, so $z = \frac{2}{\sqrt{3}} \in (0, 2)$.

Definition 2.6 If f is continuous on the interval $[a, b]$, then the average value f_{av} of f on $[a, b]$ is

$$f_{av} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Example 2.17 Find the average value of the function f on the given interval.

(1) $f(x) = x^3 + x - 1, \quad [0, 2]$

(2) $f(x) = \sqrt{x}, \quad [1, 3]$

Solution:

$$(1) \quad f_{av} = \frac{1}{2-0} \int_0^2 (x^3 + x - 1) dx = \frac{1}{2} \left[\frac{x^4}{4} + \frac{x^2}{2} - x \right]_0^2 = \frac{1}{2} [(4 + 2 - 2) - (0)] = 2.$$

$$(2) \quad f_{av} = \frac{1}{3-1} \int_1^3 \sqrt{x} dx = \frac{1}{2} \frac{2}{3} \left[x^{\frac{3}{2}} \right]_1^3 = \frac{3\sqrt{3}-1}{3}.$$

From the Fundamental Theorem, if f is continuous on $[a, b]$ and $F(x) = \int_c^x f(t) dt$ where $c \in [a, b]$, then

$$\frac{d}{dx} \int_a^x f(t) dt = \frac{d}{dx} [F(x) - F(a)] = f(x) \quad \forall x \in [a, b].$$

This result can be generalized as follows:

Theorem 2.10 Let f be continuous on $[a, b]$. If g and h are in the domain of f and differentiable, then

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x) \quad \forall x \in [a, b].$$

Corollary 2.2 Let f be continuous on $[a, b]$. If g and h are in the domain of f and differentiable, then

$$1. \quad \frac{d}{dx} \int_a^{h(x)} f(t) dt = f(h(x))h'(x) \quad \forall x \in [a, b],$$

$$2. \quad \frac{d}{dx} \int_{g(x)}^a f(t) dt = -f(g(x))g'(x) \quad \forall x \in [a, b].$$

Example 2.18 Find the derivative.

$$(1) \quad \frac{d}{dx} \int_1^x \sqrt{\cos t} \, dt$$

$$(2) \quad \frac{d}{dx} \int_1^{x^2} \frac{1}{t^3 + 1} \, dt$$

$$(3) \quad \frac{d}{dx} \left(x \int_x^{x^2} (t^3 - 1) \, dt \right)$$

$$(4) \quad \frac{d}{dx} \int_{x+1}^3 \sqrt{t+1} \, dt$$

Solution:

$$(1) \quad \frac{d}{dx} \int_1^x \sqrt{\cos t} \, dt = \sqrt{\cos x} (1) = \sqrt{\cos x}.$$

$$(2) \quad \frac{d}{dx} \int_1^{x^2} \frac{1}{t^3 + 1} \, dt = \frac{1}{(x^2)^3 + 1} (2x) = \frac{2x}{x^6 + 1}.$$

$$(3) \quad \frac{d}{dx} \left(x \int_x^{x^2} (t^3 - 1) \, dt \right) = \int_x^{x^2} (t^3 - 1) \, dt + x(2x(x^6 - 1) - (x^3 - 1))$$

(Let $f(x) = x$ and $g(x) = \int_x^{x^2} (t^3 - 1) \, dt$. Then, find $\frac{d}{dx}(fg)(x)$)

$$(4) \quad \frac{d}{dx} \int_{x+1}^3 \sqrt{t+1} \, dt = 0 - \sqrt{(x+1)+1} = -\sqrt{x+2}.$$