



CHAPTER

3

VECTORS

Physics Department

Term 1902

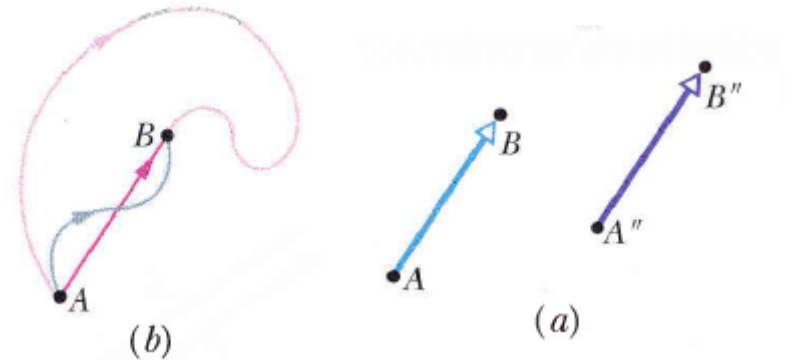
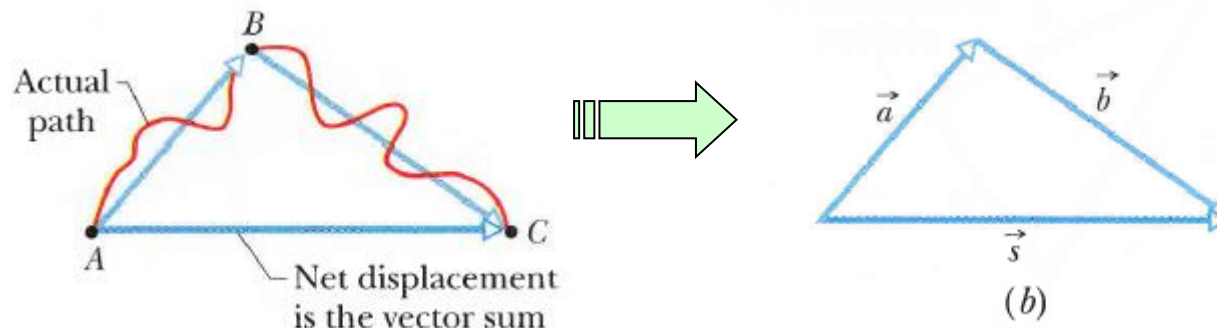
3 Vectors

3-2 | Vectors and Scalars

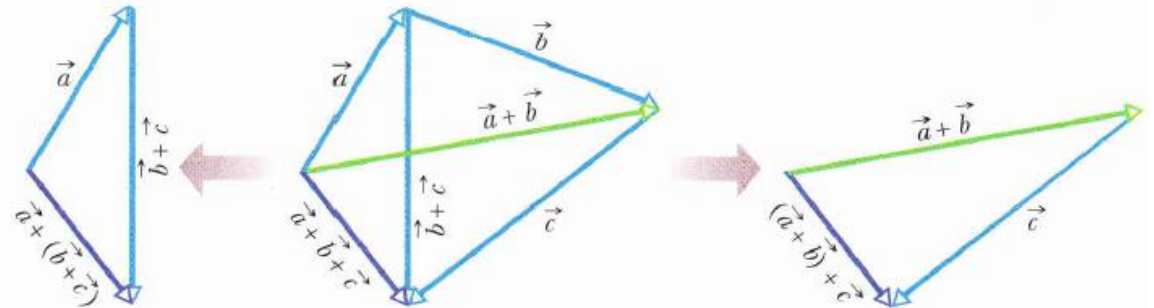
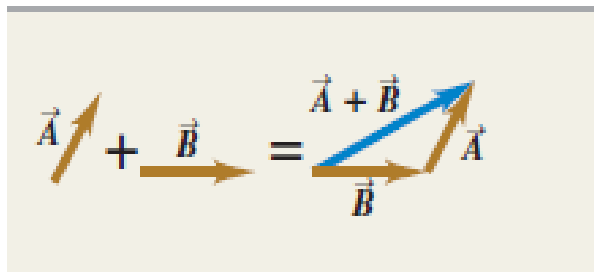
A particle moving along a straight line can move in only two directions. We can take its motion to be positive in one of these directions and negative in the other. For a particle moving in three dimensions, however, a plus sign or minus sign is no longer enough to indicate a direction. Instead, we must use a *vector*.

A **vector** has magnitude as well as direction, and vectors follow certain (vector) rules of combination, which we examine in this chapter. A **vector quantity** is a quantity that has both a magnitude and a direction

Not all physical quantities involve a direction. Temperature, pressure, energy, mass, and time, for example, do not “point” in the spatial sense. We call such quantities **scalars**,



3-3 Adding Vectors Geometrically



The three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be grouped in any way as they are added;

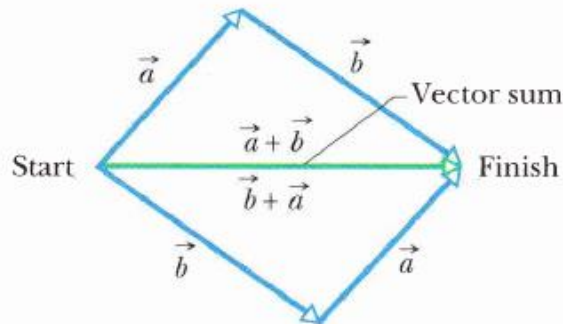
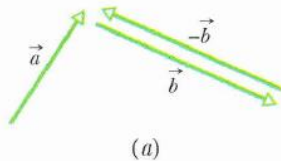
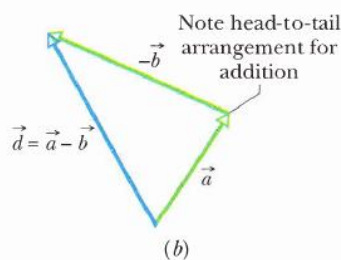


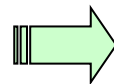
FIG. The two vectors $\vec{a}$ and $\vec{b}$ can be added in either order; see Eq. 3-2.



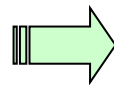
(a)



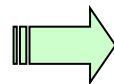
(b)



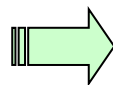
$$\vec{s} = \vec{a} + \vec{b},$$



$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \quad (\text{commutative law}).$$



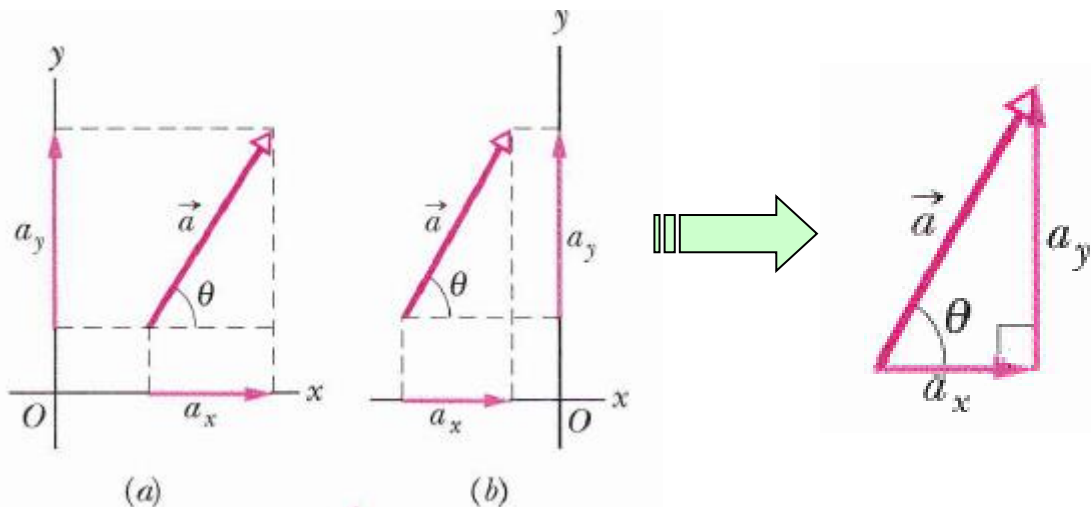
$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \quad (\text{associative law}).$$



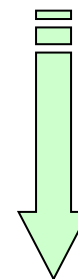
$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b}) \quad (\text{vector subtraction});$$

3-4 Components of Vectors

A **component** of a vector is the projection of the vector on an axis.



$$a_x = a \cos \theta \quad \text{and} \quad a_y = a \sin \theta,$$



$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan \theta = \frac{a_y}{a_x}$$

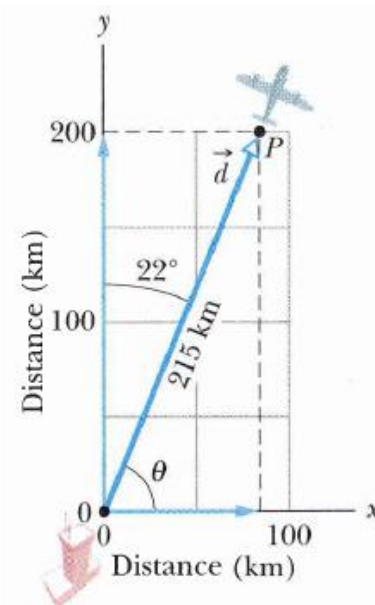
If we know a vector in *component notation* (a_x and a_y) and want it in *magnitude-angle notation* (a and θ), we can use the equations

Sample Problem

A small airplane leaves an airport on an overcast day and is later sighted 215 km away, in a direction making an angle of 22° east of due north. How far east and north is the airplane from the airport when sighted?

$$\theta = 68^\circ (= 90^\circ - 22^\circ) \Rightarrow \begin{aligned} d_x &= d \cos \theta = (215 \text{ km})(\cos 68^\circ) = 81 \text{ km} \\ d_y &= d \sin \theta = (215 \text{ km})(\sin 68^\circ) = 199 \text{ km} \end{aligned}$$

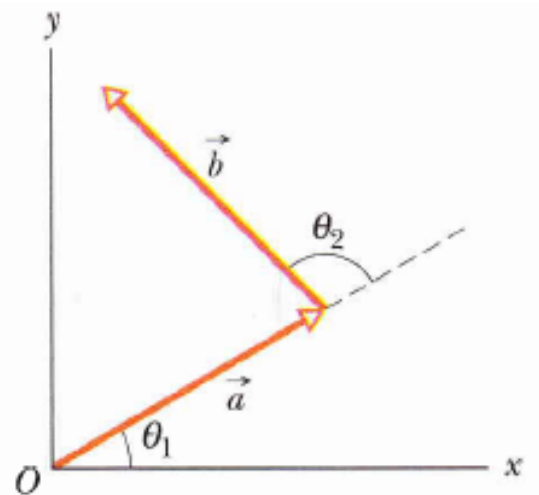
Thus, the airplane is 81 km east and 2.0×10^2 km north of the airport.



Problem 1:

The two vectors $\vec{a}$ and $\vec{b}$ in the figure below have equal magnitudes of $10m$ and the angles are $\theta_1 = 30^\circ$ and $\theta_2 = 105^\circ$. Find:

1. the x and y components of $\vec{a}$ and $\vec{b}$.
2. the x and y components of the vector sum $\vec{r} = \vec{a} + \vec{b}$
3. the magnitude of $\vec{r}$.
4. the angle $\vec{r}$ makes with the positive direction of the x axis.



3-5 | Unit Vectors

A unit vector is a dimensionless vector. We use $\hat{i}$, $\hat{j}$ and $\hat{k}$ symbols to represent the unit vectors pointing in the positive directions of x, y, and z axes, respectively. These unit vectors are perpendicular to each other and have equal magnitudes of one i.e. $i = j = k = 1$.

A **unit vector** is a vector that has a magnitude of exactly 1

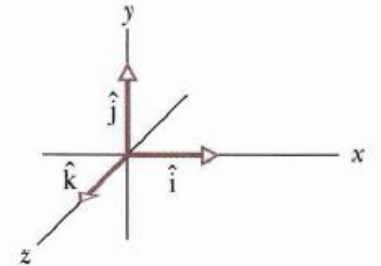
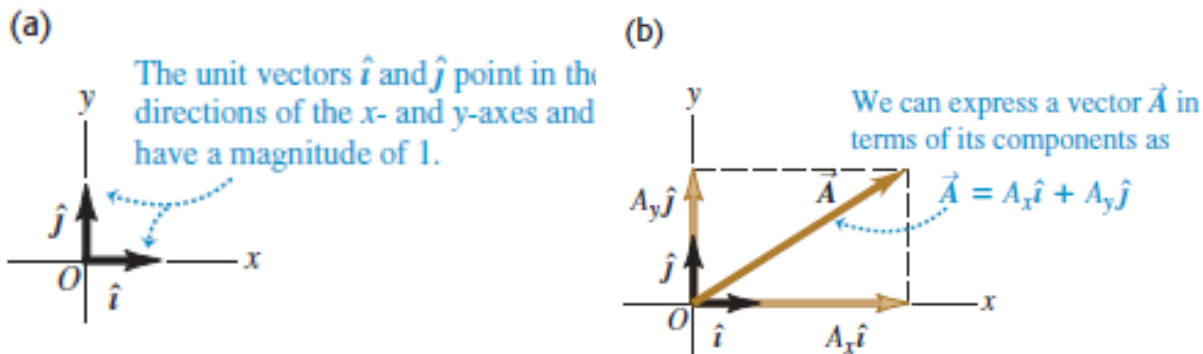


FIG. Unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ define the directions of a right-handed coordinate system.



3-6 Adding Vectors by Components

Vectors are often expressed in terms of unit vectors. Thus, the vectors $\vec{A}$ and $\vec{B}$ can be

written as: $\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$ or $\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}_{(\vec{i}, \vec{j}, \vec{k})}$

$$\vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k} \quad \text{or} \quad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{(\vec{i}, \vec{j}, \vec{k})}$$

Now, suppose we wish to add the vector $\vec{A}$ to the vector $\vec{B}$, we obtain:

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x) \vec{i} + (A_y + B_y) \vec{j} + (A_z + B_z) \vec{k} = R_x \vec{i} + R_y \vec{j} + R_z \vec{k}$$

$$\text{Or } \vec{R} = \vec{A} + \vec{B} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}_{(\vec{i}, \vec{j}, \vec{k})} + \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{(\vec{i}, \vec{j}, \vec{k})} = \begin{pmatrix} A_x + B_x \\ A_y + B_y \\ A_z + B_z \end{pmatrix}_{(\vec{i}, \vec{j}, \vec{k})}$$

Sample Problem 3-4

Figure 3-16a shows the following three vectors:

$$\vec{a} = (4.2 \text{ m})\vec{i} - (1.5\text{m})\vec{j}, \quad \vec{b} = (-1.6 \text{ m})\vec{i} + (2.9\text{m})\vec{j}, \text{ and } \vec{c} = (-3.7\text{m})\vec{j}$$

What is their vector sum $\vec{r}$ which is also shown?

Calculations: The x component of the vector sum $\vec{r}$ is:

$$r_x = a_x + b_x + c_x = 4.2\text{m} - 1.6\text{m} + 0 = 2.6\text{m}$$

$$r_y = a_y + b_y + c_y = -1.5\text{m} + 2.9\text{m} - 3.7\text{m} = -2.3\text{m}$$

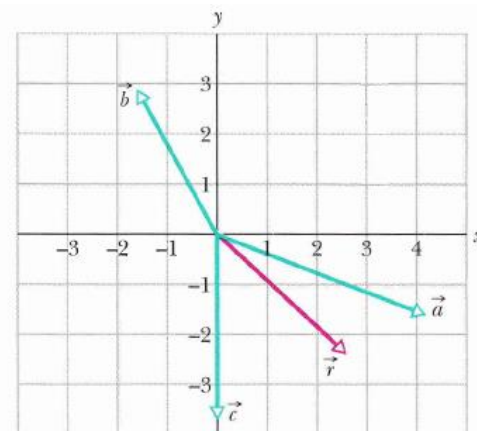
Then, the vector sum $\vec{r}$ is given by: $\vec{r} = (2.6\text{m})\vec{i} - (2.3\text{m})\vec{j}$

We can determine the magnitude and the angle for the vector $\vec{r}$:

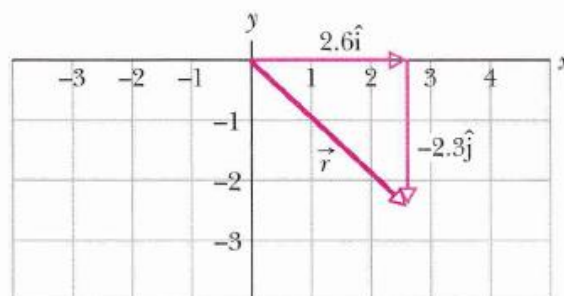
❖ The magnitude is: $r = \sqrt{(2.6)^2 + (-2.3)^2} \approx 3.5\text{m}$

❖ The angle θ (measured from +x direction) is:

$$\theta = \tan^{-1}\left(\frac{r_y}{r_x}\right) = \tan^{-1}\left(\frac{-2.3}{2.6}\right) = -41^\circ$$



(a)



(b)

FIG. 3-16 Vector $\vec{r}$ is the vector sum of the other three vectors.

3-8 Multiplying Vectors

Multiplying a Vector by a Scalar

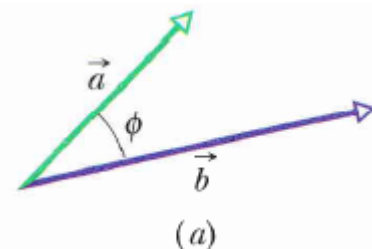
If we multiply a vector $\vec{a}$ by a scalar s , we get a new vector. Its magnitude is the product of the magnitude of $\vec{a}$ and the absolute value of s . Its direction is the direction of $\vec{a}$ if s is positive but the opposite direction if s is negative. To divide $\vec{a}$ by s , we multiply $\vec{a}$ by $1/s$.

Multiplying a Vector by a Vector

The Scalar Product

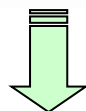
The **scalar product** of the vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = ab \cos \phi,$$

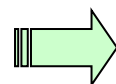


➡ If the angle ϕ between two vectors is 0° , the component of one vector along the other is maximum, and so also is the dot product of the vectors. If, instead, ϕ is 90° , the component of one vector along the other is zero, and so is the dot product.

The commutative law applies to a scalar product $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.



$$\vec{a} \cdot \vec{b} = (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \cdot (b_x \hat{i} + b_y \hat{j} + b_z \hat{k}),$$

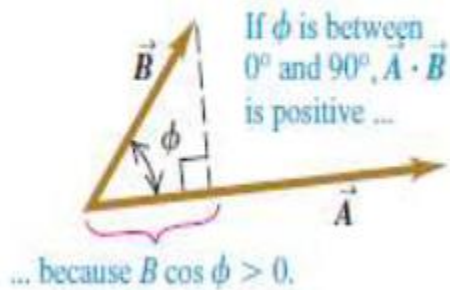


$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z.$$

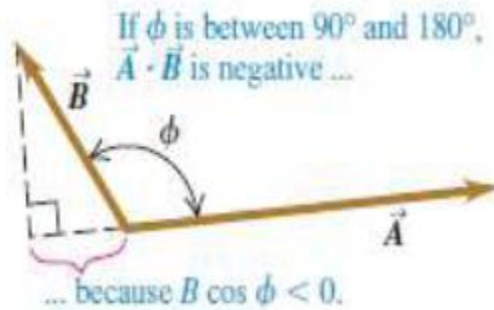
$$\vec{A} \cdot \vec{B} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}_{(\vec{i}, \vec{j}, \vec{k})} \cdot \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{(\vec{i}, \vec{j}, \vec{k})} = A_x B_x + A_y B_y + A_z B_z = AB \cos \theta$$

Where θ is the angle between the two vectors $\vec{A}$ and $\vec{B}$. θ is given by: $\cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB}$

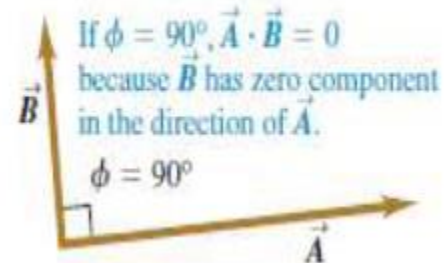
(a)



(b)



(c)



Sample Problem 3-7

What is the angle ϕ between $\vec{a} = 3.0\hat{i} - 4.0\hat{j}$ and $\vec{b} = -2.0\hat{i} + 3.0\hat{k}$? (*Caution:* Although many of the following steps can be bypassed with a vector-capable calculator, you will learn more about scalar products if, at least here, you use these steps.)

KEY IDEA

The angle between the directions of two vectors is included in the definition of their scalar product (Eq. 3-20):

$$\vec{a} \cdot \vec{b} = ab \cos \phi. \quad (3-24)$$

$$a = \sqrt{3.0^2 + (-4.0)^2} = 5.00, \quad (3-25)$$

and b is the magnitude of $\vec{b}$, or

$$b = \sqrt{(-2.0)^2 + 3.0^2} = 3.61. \quad (3-26)$$

We can separately evaluate the left side of Eq. 3-24 by writing the vectors in unit-vector notation and using the distributive law:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (3.0\hat{i} - 4.0\hat{j}) \cdot (-2.0\hat{i} + 3.0\hat{k}) \\ &= (3.0\hat{i}) \cdot (-2.0\hat{i}) + (3.0\hat{i}) \cdot (3.0\hat{k}) \\ &\quad + (-4.0\hat{j}) \cdot (-2.0\hat{i}) + (-4.0\hat{j}) \cdot (3.0\hat{k}). \end{aligned}$$

We next apply Eq. 3-20 to each term in this last expression. The angle between the unit vectors in the first term ($\hat{i}$ and $\hat{i}$) is 0° , and in the other terms it is 90° . We then have

$$\begin{aligned} \vec{a} \cdot \vec{b} &= -(6.0)(1) + (9.0)(0) + (8.0)(0) - (12)(0) \\ &= -6.0. \end{aligned}$$

Substituting this result and the results of Eqs. 3-25 and 3-26 into Eq. 3-24 yields

$$-6.0 = (5.00)(3.61) \cos \phi,$$

$$\text{so } \phi = \cos^{-1} \frac{-6.0}{(5.00)(3.61)} = 109^\circ \approx 110^\circ. \quad (\text{Answer})$$

The Vector Product

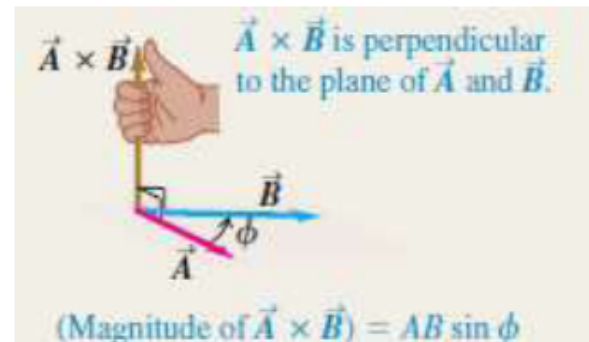
Vector product or cross product: The vector product of two vectors $\vec{A}$ and $\vec{B}$ gives a vector quantity.

$$\vec{A} \times \vec{B} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}_{(i,j,k)} \times \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{(i,j,k)} = \begin{pmatrix} A_y B_z - B_y A_z \\ A_z B_x - B_z A_x \\ A_x B_y - B_x A_y \end{pmatrix}_{(i,j,k)}$$

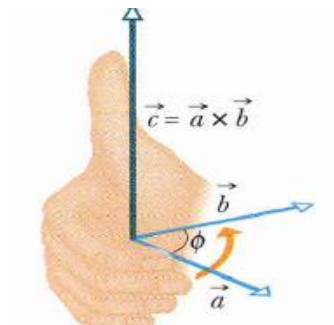
The magnitude of $\vec{A} \times \vec{B}$ is:

$$\|\vec{A} \times \vec{B}\| = \sqrt{(A_y B_z - B_y A_z)^2 + (A_z B_x - B_z A_x)^2 + (A_x B_y - B_x A_y)^2}$$

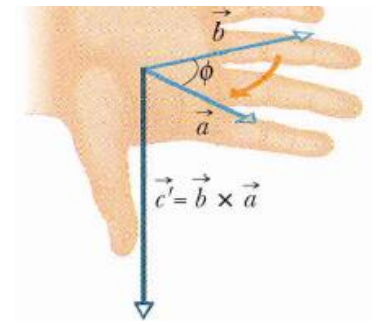
or $\|\vec{A} \times \vec{B}\| = AB \sin \theta$



When two vectors $\vec{A}$ and $\vec{B}$ are parallel or antiparallel, $\Phi=0$ or 180° , the cross product $\vec{A} \times \vec{B}$ is always zero. In particular, the vector product of any vector with itself is zero.



$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}).$$



Sample Problem 3-9

If $\vec{a} = 3\hat{i} - 4\hat{j}$ and $\vec{b} = -2\hat{i} + 3\hat{k}$, what is $\vec{c} = \vec{a} \times \vec{b}$?

KEY IDEA

When two vectors are in unit-vector notation, we can find their cross product by using the distributive law.

Calculations: Here we write

$$\begin{aligned}\vec{c} &= (3\hat{i} - 4\hat{j}) \times (-2\hat{i} + 3\hat{k}) \\ &= 3\hat{i} \times (-2\hat{i}) + 3\hat{i} \times 3\hat{k} + (-4\hat{j}) \times (-2\hat{i}) \\ &\quad + (-4\hat{j}) \times 3\hat{k}.\end{aligned}$$

We next evaluate each term with Eq. 3-27, finding the direction with the right-hand rule. For the first term here, the angle ϕ between the two vectors being crossed is 0 . For the other terms, ϕ is 90° . We find

$$\begin{aligned}\vec{c} &= -6(0) + 9(-\hat{j}) + 8(-\hat{k}) - 12\hat{i} \\ &= -12\hat{i} - 9\hat{j} - 8\hat{k}.\end{aligned}\quad (\text{Answer})$$

This vector $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, a fact you can check by showing that $\vec{c} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{b} = 0$; that is, there is no component of $\vec{c}$ along the direction of either $\vec{a}$ or $\vec{b}$.

Problem 2:

Two vectors $\vec{A}$ and $\vec{B}$ are given by the following expressions:

$$\vec{A} = 4\hat{i} + 3\hat{j} - 2\hat{k} \quad \& \quad \vec{B} = 3\hat{i} - 2\hat{j} - 5\hat{k}$$

Find out:

1. their magnitudes $\|\vec{A}\|$ and $\|\vec{B}\|$.
2. their resultant $\vec{R} = \vec{A} + \vec{B}$.
3. their scalar product $\vec{A} \cdot \vec{B}$.
4. their vector product $\vec{A} \times \vec{B}$.
5. the angle θ between the two vectors.

Suggestion problems and Simple Problems

Simple Problems:

3-4, 3-7, 3-9

Problems:

6, 12, 41, 43

REVIEW & SUMMARY

Scalars and Vectors *Scalars*, such as temperature, have magnitude only. They are specified by a number with a unit (10°C) and obey the rules of arithmetic and ordinary algebra. *Vectors*, such as displacement, have both magnitude and direction (5 m, north) and obey the rules of vector algebra.

Adding Vectors Geometrically Two vectors $\vec{a}$ and $\vec{b}$ may be added geometrically by drawing them to a common scale and placing them head to tail. The vector connecting the tail of the first to the head of the second is the vector sum $\vec{r}$. To subtract $\vec{b}$ from $\vec{a}$, reverse the direction of $\vec{b}$ to get $-\vec{b}$; then add $-\vec{b}$ to $\vec{a}$. Vector addition is commutative and obeys the associative law.

Components of a Vector The (scalar) *components* a_x and a_y of any two-dimensional vector $\vec{a}$ along the coordinate axes are found by dropping perpendicular lines from the ends of $\vec{a}$ onto the coordinate axes. The components are given by

$$a_x = a \cos \theta \quad \text{and} \quad a_y = a \sin \theta, \quad (3-5)$$

where θ is the angle between the positive direction of the x axis and the direction of $\vec{a}$. The algebraic sign of a component indicates its direction along the associated axis. Given its components, we can find the magnitude and orientation of the vector $\vec{a}$ with

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan \theta = \frac{a_y}{a_x}. \quad (3-6)$$

Unit-Vector Notation *Unit vectors* $\hat{i}$, $\hat{j}$, and $\hat{k}$ have magnitudes of unity and are directed in the positive directions of the x , y , and z axes, respectively, in a right-handed coordinate system. We can write a vector $\vec{a}$ in terms of unit vectors as

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}, \quad (3-7)$$

in which $a_x \hat{i}$, $a_y \hat{j}$, and $a_z \hat{k}$ are the **vector components** of $\vec{a}$ and a_x , a_y , and a_z are its **scalar components**.

Adding Vectors in Component Form To add vectors

in component form, we use the rules

$$r_x = a_x + b_x \quad r_y = a_y + b_y \quad r_z = a_z + b_z. \quad (3-11 \text{ to } 3-13)$$

Here $\vec{a}$ and $\vec{b}$ are the vectors to be added, and $\vec{r}$ is the vector sum.

Product of a Scalar and a Vector The product of a scalar s and a vector $\vec{v}$ is a new vector whose magnitude is sv and whose direction is the same as that of $\vec{v}$ if s is positive, and opposite that of $\vec{v}$ if s is negative. To divide $\vec{v}$ by s , multiply $\vec{v}$ by $1/s$.

The Scalar Product The **scalar** (or **dot**) **product** of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a} \cdot \vec{b}$ and is the *scalar* quantity given by

$$\vec{a} \cdot \vec{b} = ab \cos \phi, \quad (3-20)$$

in which ϕ is the angle between the directions of $\vec{a}$ and $\vec{b}$. A scalar product is the product of the magnitude of one vector and the scalar component of the second vector along the direction of the first vector. In unit-vector notation,

$$\vec{a} \cdot \vec{b} = (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \cdot (b_x \hat{i} + b_y \hat{j} + b_z \hat{k}), \quad (3-22)$$

which may be expanded according to the distributive law. Note that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

The Vector Product The **vector** (or **cross**) **product** of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a} \times \vec{b}$ and is a *vector* $\vec{c}$ whose magnitude c is given by

$$c = ab \sin \phi, \quad (3-27)$$

in which ϕ is the smaller of the angles between the directions of $\vec{a}$ and $\vec{b}$. The direction of $\vec{c}$ is perpendicular to the plane defined by $\vec{a}$ and $\vec{b}$ and is given by a right-hand rule, as shown in Fig. 3-21. Note that $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$. In unit-vector notation,

$$\vec{a} \times \vec{b} = (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \times (b_x \hat{i} + b_y \hat{j} + b_z \hat{k}), \quad (3-29)$$

which we may expand with the distributive law.