

Chapter I: The Riemann integrals

1-Riemann sums

Riemann was led to the development of the integral by trying to characterize which functions were integrable according to Cauchy's definition.

In the process, he modified Cauchy's definition and developed the theory of integration which now bears his name. One of his achievements was that he was able to provide necessary and sufficient conditions for a real-valued bounded function to be integrable.

Upper and Lower Sums:

Let $I = [a, b]$, $a < b$, be a given closed and bounded interval in $\mathbb{R}$. A partition P of $[a, b]$ is specified by a sequence $P = \{x_0, x_1, \dots, x_n\}$ of points $x_0, x_1, \dots, x_n \in [a, b]$ with $a = x_0 < x_1 < x_2 \dots x_{n-1} < x_n = b$, here $n \in \{1, 2, 3, \dots\}$.

Then, $[a, b] = \bigcup_{k=1}^n [x_{k-1}, x_k]$.

Definition: (Uniform partition)

Given a partition $P = \{a = x_0, x_1, \dots, x_n = b\}$, P is called a uniform partition of $[a, b]$ if $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \dots = x_n - x_{n-1} = \frac{b-a}{n}$.

Example: The uniform partition of $[1, 5]$ is $P = \{1, 2, 3, 4, 5\}$.

Set $\Delta x_k = x_k - x_{k-1}$, which is equal to the length of the interval $[x_{k-1}, x_k]$.

Suppose $f: [a, b] \rightarrow \mathbb{R}$ is a bounded real-valued function.

For each $k = 1, 2, 3, \dots, n$, let

$$M_k = \sup_{t \in [x_{k-1}, x_k]} f(t), \quad m_k = \inf_{t \in [x_{k-1}, x_k]} f(t).$$

$$M_k = \sup\{f(t): x_{k-1} \leq t \leq x_k\}, \quad m_k = \inf\{f(t): x_{k-1} \leq t \leq x_k\}.$$

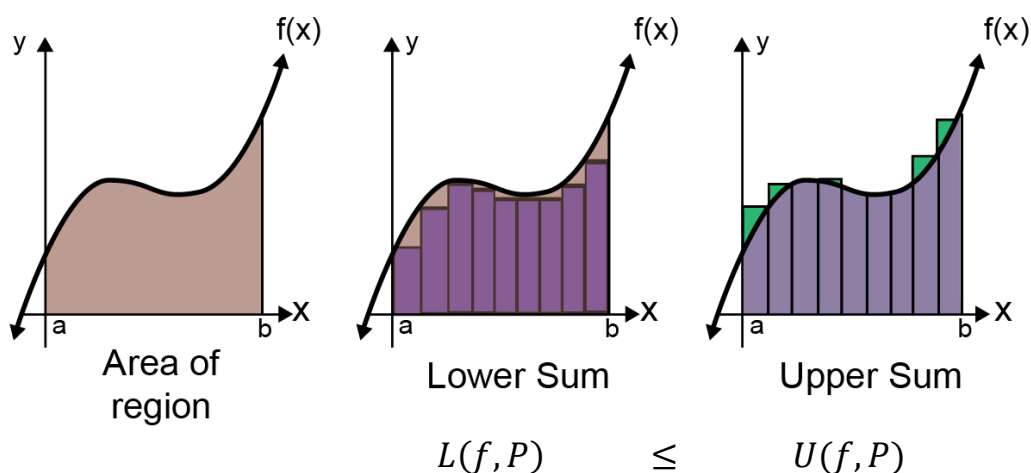
The upper Riemann sum $U(f, P)$ is defined by:

$$U(f, P) = \sum_{k=1}^n M_k \Delta x_k.$$

Similarly, the lower Riemann sum $L(f, P)$ is defined by:

$$L(f, P) = \sum_{k=1}^n m_k \Delta x_k.$$

Remark: Since $m_k \leq M_k$ for all $k = 1, 2, 3, \dots, n$, for every partition P , we always have $L(P, f) \leq U(P, f)$.



▪ Upper and Lower Integrals:

Let $I = [a, b]$ be a given closed and bounded interval in $\mathbb{R}$ and let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function.

If the function f satisfies $m \leq f(t) \leq M$, for all $t \in [a, b]$, then we have

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a),$$

for any partition P of $[a, b]$. Since $M_k \leq M$ for all $k = 1, 2, 3, \dots, n$,

$$U(f, P) = \sum_{k=1}^n M_k \Delta x_k \leq \sum_{k=1}^n M(x_k - x_{k-1}) \leq M(b-a).$$

Similarly, $L(P, f) \geq m(b-a)$.

Definition:(Upper and Lower Integrals)

Let $I = [a, b]$ be a given closed and bounded interval in $\mathbb{R}$ and let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function. Then, the upper integral of f on I is defined by:

$$U(f) = \int_a^b f(x) dx = \inf_{P \in P(I)} U(f, P),$$

Similarly, the lower integral of f on I is defined by:

$$L(f) = \int_a^b f(x) dx = \sup_{P \in P(I)} L(f, P).$$

Lemma: Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded real-valued function on $[a, b]$. Then,

$$L(f) = \int_a^b f(x) dx \leq U(f) = \int_a^b f(x) dx.$$

2- The Riemann integral

If $f: [a, b] \rightarrow \mathbb{R}$ is bounded, then the lower and upper integrals of f on $[a, b]$ always exist and satisfy $L(f) \leq U(f)$.

Definition:(Riemann integral)

Let $I = [a, b]$ and let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded real-valued function on $[a, b]$. Then, f is said to be **Riemann integral** on I if $L(f) = U(f) = A$, or

$$\int_a^b f(x) dx = \int_a^b f(x) dx$$

and this number is denoted by $A = \int_a^b f(x) dx$ or $A = \int_a^b f$.

We denote by $\mathcal{R}([a, b])$, the set of all **Riemann integral** functions on $[a, b]$.

In addition, if $f \in \mathcal{R}([a, b])$, we define

$$\int_a^b f = - \int_b^a f.$$

If $f: [a, b] \rightarrow \mathbb{R}$ satisfies $m \leq f(x) \leq M$ for all $t \in [a, b]$ then,

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

If in addition, $f \in \mathcal{R}([a, b])$, then

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

If $f(x) \geq 0$, for all $x \in [a, b]$, then $\int_a^b f \geq 0$.

If $f \in \mathcal{R}([a, b])$, is non-negative, then the quantity $\int_a^b f$ represents the area of the region bounded above by the graph $y = f(x)$, below by the x -axis, and the lines $x = a$ and $x = b$.

Example of non-integrable functions:

Example:

Let $I = [a, b]$, and let the Dirichlet function $D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$. Prove that $D(x)$ is not Riemann integrable.

Proof. Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be partition which divide $[a, b]$, $a < b$.

Since $m_k = 0$ $M_k = 1$, then

$$L(P, D) = \sum_{k=1}^n 0 (x_k - x_{k-1}) = 0,$$

$$\Delta x = \frac{b-a}{n} = x_k - x_{k-1}$$

$$U(P, D) = \sum_{k=1}^n (x_k - x_{k-1}) = (x_n - x_0) = (b - a).$$

Since P is any partition then,

$$L(D) = \int_a^b f dx = 0, \quad U(D) = \int_a^b f dx = b - a.$$

That is $L(D) \neq U(D)$ which means that the function $D(x)$ is not integrable.

Example of integrable functions:Example:

Let $f(x) = c$ be constant function on $I = [a, b]$, then f is Riemann integrable and

$$\int_a^b c \, dx = c(b - a).$$

Proof. Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be any partition of I . Since $m_k = M_k = c$, then

$$\begin{aligned} L(P, f) &= \sum_{k=1}^n m_k \Delta x_k = c \sum_{k=1}^n (x_k - x_{k-1}) \\ &= c [(x_1 - x_0) + (x_2 - x_1) + \dots + (x_n - x_{n-1})] \\ &= c (x_n - x_0) = c(b - a). \end{aligned}$$

Similarly, $U(P, f) = \sum_{k=1}^n M_k \Delta x_k = c(b - a)$. Consequently, $f(x) = c$ is integrable on $[a, b]$ and

$$\int_a^b f \, dx = \int_a^b f \, dx = \int_a^b c \, dx = c(b - a).$$

Example:

Let $f(x) = x$ on $I = [0, 1]$, then f is Riemann integrable and $\int_0^1 x \, dx = \frac{1}{2}$.

Proof. Let P_n be partition which divide $[a, b]$ into n equal subintervals given by

$$x_k = a + \frac{k}{n}(b - a), \quad \forall 0 \leq k \leq n.$$

For $I = [0, 1]$, $x_k = \frac{k}{n}$, $\forall 0 \leq k \leq n$. Then, $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$.

Since $f(x)$ is increasing, so on each interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, we have

$$m_k = \frac{k-1}{n}, \quad M_k = \frac{k}{n}$$

Moreover, $\Delta x_k = x_k - x_{k-1} = \frac{k}{n} - \frac{k-1}{n} = \frac{1}{n}$.

Recall that, $\sum_{k=1}^n (k-1) = \left(\frac{n(n-1)}{2}\right) = \left(\frac{n^2-n}{2}\right)$ and $\sum_{k=1}^n k = \left(\frac{n(n+1)}{2}\right) = \left(\frac{n^2+n}{2}\right)$.

Now we find,

$$L(P_n, f) = \sum_{k=1}^n m_k \Delta x_k = \sum_{k=1}^n x_{k-1} \Delta x_k = \sum_{k=1}^n \frac{k-1}{n} \left(\frac{1}{n}\right) = \frac{1}{n^2} \sum_{k=1}^n (k-1);$$

$$L(P_n, f) = \frac{1}{n^2} \left(\frac{n^2-n}{2}\right) = \left(\frac{1}{2} - \frac{1}{2n}\right),$$

$$U(P_n, f) = \sum_{k=1}^n M_k \Delta x_k = \sum_{k=1}^n x_k \Delta x_k = \left(\frac{1}{2} + \frac{1}{2n}\right).$$

Thus, $\sup_n L(P_n, f) = \frac{1}{2}$ and $\inf_n U(P_n, f) = \frac{1}{2}$.

Since the collection $\{P_n : n \in \mathbb{N}\}$ is a subset of the set of all partitions of $[0,1]$,

$$\frac{1}{2} = \sup_n L(P_n, f) \leq \sup_n L(P, f) = L(f),$$

$$\frac{1}{2} = \inf_n U(P_n, f) \leq \inf_n U(P, f) = U(f).$$

Therefore $f(x) = x$ is integrable on $[0,1]$ with $\int_0^1 x \, dx = \frac{1}{2}$.

Example:

Let $f(x) = x^2$ on $I = [0,1]$, then f is Riemann integrable and $\int_0^1 x^2 \, dx = \frac{1}{3}$.

Proof. Let P_n be partition which divide $[0,1]$ into n equal subintervals given by

$$x_k = a + \frac{k}{n}(b-a), \quad \forall 0 \leq k \leq n.$$

For $I = [0,1]$, $x_k = \frac{k}{n}$, $\forall 0 \leq k \leq n$. Then, $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$.

Since $f(x)$ is increasing on $[0,1]$, so on each interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, we have

$$m_k = \left(\frac{k-1}{n}\right)^2, \quad M_k = \left(\frac{k}{n}\right)^2$$

Moreover, $\Delta x_k = x_k - x_{k-1} = \frac{k}{n} - \frac{k-1}{n} = \frac{1}{n}$.

Recall that, $\sum_{k=1}^n (k-1)^2 = \sum_{k=1}^n (k-1)^2 = \frac{1}{6} (n-1)n(2n-1) = \frac{1}{6} (2n^2 - 3n + 1)$ and

$$\sum_{k=1}^n k^2 = \frac{1}{6} n(n+1)(2n+1) = \frac{1}{6} (2n^2 + 3n + 1).$$

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Now we find,

Low sum

$$L(P_n, f) = \sum_{k=1}^n m_k \Delta x_k = \sum_{k=1}^n \left(\frac{k-1}{n} \right)^2 \left(\frac{1}{n} \right) = \frac{1}{n^3} \sum_{k=1}^n (k-1)^2;$$

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$$L(P_n, f) = \frac{1}{6} \left(\frac{n(n-1)(2n-1)}{n^3} \right) = \frac{1}{3} \left(1 - \frac{1}{n} \right) \left(1 - \frac{1}{2n} \right),$$

upper sum

$$U(P_n, f) = \frac{1}{6} \left(\frac{n(n+1)(2n+1)}{n^3} \right) = \frac{1}{3} \left(1 + \frac{1}{n} \right) \left(1 + \frac{1}{2n} \right).$$

Thus, $\sup_n L(P_n, f) = \frac{1}{3}$ and $\inf_n U(P_n, f) = \frac{1}{3}$.

Since the collection $\{P_n : n \in \mathbb{N}\}$ is a subset of the set of all partitions of $[0, 1]$,

$$\frac{1}{3} = \sup_n L(P_n, f) \leq \sup_n L(P, f) = L(f),$$

$$\frac{1}{3} = \inf_n U(P_n, f) \leq \inf_n U(P, f) = U(f).$$

Therefore $f(x) = x^2$ is integrable on $[0, 1]$ with $\int_0^1 x^2 dx = \frac{1}{3}$.

Exercises:

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1- Let $f(x) = 1 - x^2$ on $[-1, 2]$, then find $L(P, f)$ and $U(P, f)$ for a partition

$P = \{-1, 0, 1, 2\}$ of $[-1, 2]$.

2- Let $f(x) = 1 - x^2$ on $[-1, 2]$, then find $L(P, f)$ and $U(P, f)$ for a partition

$P = \left\{ -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \right\}$ of $[-1, 2]$.

3- If $f(x) = 1 - x^2$ on $I = [0, 1]$, prove that f is integrable and find $\int_0^1 f dx$.

4- If $h(x) = 3x + 1$ on $I = [0, 1]$, prove that h is integrable and find $\int_0^1 h dx$.

3-Properties of the Riemann Integral

Theorem:

Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded function on $[a, b]$. Then, $f(x)$ is integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there is a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

Theorem:

Let f be a real-valued function on $[a, b]$.

(I) If f is continuous on $[a, b]$, then $f \in \mathcal{R}([a, b])$.

(II) If f is monotone (increasing or decreasing) on $[a, b]$, then $f \in \mathcal{R}([a, b])$.

Theorem:

Let $f, g: [a, b] \rightarrow \mathbb{R}$ be integrable functions on $[a, b]$; $f, g \in \mathcal{R}([a, b])$. Then,

(I) $kf \in \mathcal{R}([a, b])$, for all $k \in \mathbb{R}$.

(II) $f + g \in \mathcal{R}([a, b])$, with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

(III) $fg \in \mathcal{R}([a, b])$.

Corollary:

Let $f: [a, b] \rightarrow \mathbb{R}$ be integrable functions on $[a, b]$; $f \in \mathcal{R}([a, b])$. Then,

(I) $|f| \in \mathcal{R}([a, b])$, with $\left| \int_a^b f \right| \leq \int_a^b |f|$.

(II) $f^2 \in \mathcal{R}([a, b])$.

Theorem:

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded real-valued function on $[a, b]$, and suppose that $a < c < b$ where $c \in \mathbb{R}$. Then, $f \in \mathcal{R}([a, b])$ if and only if $f \in \mathcal{R}([a, c])$ and $f \in \mathcal{R}([c, b])$. If this is the case, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Example: Show that the following functions are integrable or not.

a) $f(x) = x^7 + 5x$ on $[-1,6]$ b) $f(x) = 1 - x^2$ on the interval $[0,2]$

c) $f(x) = \cos x + 5x + e^x$ on the interval $[0, \pi]$

Solution:

a) Since $f'(x) = 7x^6 + 5 > 0$, for all $x \in [-1,6]$, then the function $f(x) = x^7 + 5x$ is increasing, and by the previous theorem it is integrable on $[-1,6]$.

b) Since $f'(x) = -2x < 0$, for all $x \in [0,2]$, then the function $f(x) = 1 - x^2$ is decreasing, and by the previous theorem it is integrable on $[0,2]$.

c) We know from calculus that the function $f(x) = \cos x + 5x + e^x$ is continuous on the interval $[0, \pi]$, then by the previous theorem it is integrable on $[0, \pi]$.

Riemann's Definition of the Integral

Definition:

Let $f: [a, b] \rightarrow R$ be bounded function. If $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$.

Choose numbers $t_k \in [x_{k-1}, x_k]$, for $k = 1, 2, 3, \dots, n$. The sum

$$S(P, f) = \sum_{k=1}^n f(t_k) \Delta x_k$$

is called a Riemann sum for f , with respect to P and the points $\{t_k\}$.

Definition:

For a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, we set

$$\|p\| = \max_{1 \leq k \leq n} \Delta x_k = \max\{(x_k - x_{k-1}): k = 1, 2, 3, \dots, n\}.$$

The quantity $\|p\|$ is called the **norm** or the **mesh** of the partition P .

Theorem:

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded real-valued function on $[a, b]$. If Riemann sum $S(P, f) = \sum_{k=1}^n f(t_k) \Delta x_k$ exists, where $t_k \in [x_{k-1}, x_k]$. Then, $f \in \mathcal{R}([a, b])$ and

$$\lim_{\|p\| \rightarrow 0} S(P, f) = \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n f(t_k) \Delta x_k = \int_a^b f(x) dx.$$

Conversely, if $f \in \mathcal{R}([a, b])$, then $\lim_{\|p\| \rightarrow 0} S(P, f)$ exists and

$$\lim_{\|p\| \rightarrow 0} S(P, f) = \int_a^b f.$$

Remark: We can use the integral Riemann sum to prove the following property:

If $f \in \mathcal{R}([a, b])$, then

(I) $kf \in \mathcal{R}([a, b])$, for all $k \in \mathbb{R}$.

(II) $f \pm g \in \mathcal{R}([a, b])$, with $\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$.

(III) $fg \in \mathcal{R}([a, b])$.

Proof. For any partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, set $\|p\| = \max_{1 \leq k \leq n} \Delta x_k$. Let

$f, g: [a, b] \rightarrow \mathbb{R}$ be integrable functions. Then, for any $t_k \in [x_{k-1}, x_k]$

$$\begin{aligned} (I) \quad \int_a^b cf(x) dx &= \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n cf(t_k) \Delta x_k \\ &= c \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n f(t_k) \Delta x_k = c \int_a^b f(x) dx. \end{aligned}$$

(II)

$$\begin{aligned} \lim_{\|p\| \rightarrow 0} S(P, f) &= \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n f(t_k) \Delta x_k = \int_a^b f(x) dx, \\ \lim_{\|p\| \rightarrow 0} S(P, g) &= \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n g(t_k) \Delta x_k = \int_a^b g(x) dx. \end{aligned}$$

Consider

$$\begin{aligned}
 \lim_{\|p\| \rightarrow 0} S(P, f \pm g) &= \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n (f \pm g)(t_k) \Delta x_k \\
 &= \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n f(t_k) \Delta x_k \pm \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n g(t_k) \Delta x_k \\
 &= \int_a^b f(x) dx \pm \int_a^b g(x) dx.
 \end{aligned}$$

(III) Since $fg = \frac{1}{2} [(f+g)^2 - f^2 - g^2]$, then

$$\begin{aligned}
 \lim_{\|p\| \rightarrow 0} S(P, fg) &= \frac{1}{2} \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n [(f+g)^2 - f^2 - g^2](y_k) \Delta x_k \\
 &= \frac{1}{2} \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n (f+g)^2(t_k) \Delta x_k - \frac{1}{2} \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n f^2(t_k) \Delta x_k \\
 &\quad - \frac{1}{2} \lim_{\|p\| \rightarrow 0} \sum_{k=1}^n g^2(t_k) \Delta x_k \\
 &= \frac{1}{2} \int_a^b [f^2 + g^2 + 2fg - f^2 - g^2](x) dx = \int_a^b fg(x) dx.
 \end{aligned}$$

Example:

Use the method of Riemann sums to evaluate $\int_a^b x dx$.

Solution:

Let $f(x) = x$, since $f \in \mathcal{R}([a, b])$, for any partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, then

$$\int_a^b x dx = \lim_{\|p\| \rightarrow 0} S(P, x).$$

Since the limit exists for any $t_k \in [x_{k-1}, x_k]$, we can take $t_k = \frac{1}{2}(x_{k-1} + x_k)$.

With this choice of t_k , we have

$$S(P, x) = \frac{1}{2} \sum_{k=1}^n (x_{k-1} + x_k) (x_{k-1} - x_k) = \frac{1}{2} \sum_{k=1}^n (x_{k-1})^2 - (x_k)^2 = \frac{1}{2} (b^2 - a^2).$$

Example:

Use the method of Riemann sums to evaluate $\int_a^b x^2 dx$.

Solution:

Let $f(x) = x^2$, since $f \in \mathcal{R}([a, b])$, for any partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, then

$$\int_a^b x^2 dx = \lim_{\|P\| \rightarrow 0} S(P, x^2).$$

Since the limit exists for any $t_k \in [x_{k-1}, x_k]$, we can take

$$t_k = \sqrt{\frac{[(x_{k-1})^2 + (x_k)^2 + x_k x_{k-1}]}{3}}.$$

With this choice of t_k , we have

$$\begin{aligned} S(P, x^2) &= \frac{1}{3} \sum_{k=1}^n [(x_{k-1})^2 + (x_k)^2 + x_k x_{k-1}] (x_{k-1} - x_k) \\ &= \frac{1}{3} \sum_{k=1}^n (x_{k-1})^3 - (x_k)^3 = \frac{1}{3} (b^3 - a^3). \end{aligned}$$

Theorem: Let $f: [a, b] \rightarrow \mathbb{R}$ be a Riemann integral function on $[a, b]$ and

$$S_n = \frac{b-a}{n} \sum_{k=1}^n f\left(a + \frac{k(b-a)}{n}\right).$$

Then,

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x) dx.$$

Remark: If $[a, b] = [0, 1]$, then

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx.$$

Example: Write the following limits as an integral on $[0,1]$.

(Evaluate the limit $\lim_{n \rightarrow \infty} S_n$).

$$1) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k} \quad 2) \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^n k^2 \quad 3) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n}{n^2+k^2}.$$

Solution: $\sum \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$

$$\begin{aligned} 1) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + \frac{k}{n}} = \int_0^1 f\left(\frac{k}{n}\right) dx = \int_0^1 \frac{1}{1+x} dx; \\ \int_0^1 \frac{1}{1+x} dx &= [\ln(1+x)]_0^1 = \ln(2). \end{aligned}$$

$$2) \quad \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^n k^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{k^2}{n^2} = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

$$\begin{aligned} 3) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n}{n^2+k^2} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + \left(\frac{k}{n}\right)^2} = \int_0^1 \frac{1}{1+x^2} dx; \\ \int_0^1 \frac{1}{1+x^2} dx &= [\tan^{-1}(x)]_0^1 = \frac{\pi}{4}. \end{aligned}$$

Exercise:

Write the following limits as an integral. (Evaluate the limit $\lim_{n \rightarrow \infty} S_n$).

$$1) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{n^2+k^2}$$

$$2) \quad \lim_{n \rightarrow \infty} \frac{1}{n^5} \sum_{k=1}^n k^4$$

$$3) \quad \lim_{n \rightarrow \infty} \frac{1}{n^8} \sum_{k=1}^n k^7$$

$$4) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sin\left(\frac{k\pi}{2n}\right)$$

4-Fundamental Theorem of Calculus

Definition:

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Let f be a real-valued function on an interval $[a, b]$. A function F on $[a, b]$ is called an **antiderivative** of f on $[a, b]$ if $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem: Fundamental Theorem of Calculus (First Form)

Let $f: [a, b] \rightarrow \mathbb{R}$ be an integral function on $[a, b]$ and let $F: [a, b] \rightarrow \mathbb{R}$ be an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a).$$

$$\begin{aligned} \int_a^b f(x) dx &= F(b) - F(a) \\ &= F(b) - F(a) \end{aligned}$$

Proof. Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be any partition of $[a, b]$. If F is an antiderivative of f on $[a, b]$, then by the Mean Value Theorem of Calculus

$$\frac{F(b) - F(a)}{b - a} = F'(c), \quad c \in [a, b].$$

Then, for each $k = 1, \dots, n$, there exists $t_k \in (x_{k-1}, x_k)$ such that

$$F(x_k) - F(x_{k-1}) = F'(t_k)(x_k - x_{k-1}) = f(t_k)\Delta x_k.$$

Therefore,

$$\sum_{k=1}^n f(t_k)\Delta x_k = \sum_{k=1}^n F(x_k) - F(x_{k-1}) = F(b) - F(a).$$

Since,

$$L(P, f) \leq \sum_{k=1}^n f(t_k)\Delta x_k \leq U(P, f),$$

we have

$$L(P, f) \leq F(b) - F(a) \leq U(P, f).$$

Thus, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f(x) dx = F(b) - F(a).$$

Example: Find an antiderivative of the function $f(x) = x^n, n \in \mathbb{N}$.

Solution: For any $a, b \in \mathbb{R}, a < b$ we have

$$F(x) = \int_a^b x^n dx = \left. \frac{x^{n+1}}{n+1} \right|_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

Example: Consider the function F on $[0,1]$ defined by

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Find the function $f(x)$ with antiderivative $F(x)$.

Solution:

A straightforward computation gives

$$f(x) = F'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then, f is bounded on $[0,1]$ and continuous everywhere except $x = 0$.

Thus, $f \in \mathcal{R}([0,1])$, and by the previous theorem

$$\int_0^1 f = F(1) - F(0) = \sin 1.$$

Example: Find an antiderivative on $[0,2]$ of the function f is defined on $[0,2]$ by

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ x - 1, & 1 < x \leq 2. \end{cases}$$

Solution: let $F(x)$ be defined by $F(x) = \int_0^x f(t) dt$.

If $0 \leq x \leq 1$, then $F(x) = \int_0^x 1 dt = x$. On the other hand, if $1 < x \leq 2$, then

$$F(x) = \int_0^1 1 dt + \int_1^x (t - 1) dt = \frac{x^2}{2} - x + \frac{3}{2}.$$

Thus,

$$F(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ \frac{x^2}{2} - x + \frac{3}{2}, & 1 < x \leq 2. \end{cases}$$

Example: Find the function f with an antiderivative defined by $F(x) = \frac{x|x|}{2}$ on $\mathbb{R}$.

Solution: Since

$$F(x) = \begin{cases} x^2/2 & x \geq 0, \\ -x^2/2 & x < 0. \end{cases}$$

Then,

$$f(x) = F'(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Example: Find an antiderivative of the function $f(x) = x^{-4}$ on $\mathbb{R} \setminus \{0\}$.

Solution: Since $f(x) = x^{-4}$ is defined on $\mathbb{R} \setminus \{0\}$. So, the antiderivative is

$$F(x) = \begin{cases} -\frac{x^{-3}}{3} + c_1 & x > 0, & c_1 \in \mathbb{R}, \\ -\frac{x^{-3}}{3} + c_2 & x < 0, & c_2 \in \mathbb{R}. \end{cases}$$

$$\frac{1}{n^k} \quad n \neq 0$$

$$f(x) = x^{-3} = \frac{-1}{3x^2} + C \quad x \neq 0$$

Observe that $f(x) = x^{-4}$ has no antiderivative on $[-1, 1]$. Moreover, it has no antiderivative on any interval containing $x = 0$.

Example: Let f be continuous function on $[a, b]$, and let $H: [a, b] \rightarrow \mathbb{R}$ be defined by $H(x) = \int_x^b f(t)dt$ for all $x \in [a, b]$. Then, find $H'(x)$ for $x \in [a, b]$.

Solution: Since $H(x) = \int_x^b f(t)dt = - \int_b^x f(t)dt$, then $H'(x) = -f(x)$.

Theorem: The Fundamental Theorem of Calculus (Second Form)

Let $f: [a, b] \rightarrow \mathbb{R}$ be an integral function on $[a, b]$. Define F on $[a, b]$ by

$$F(x) = \int_a^x f(t)dt.$$

Then, F is continuous on $[a, b]$. Furthermore, if f is continuous at a point $c \in [a, b]$, then F is differentiable at c and

$$F'(c) = f(c).$$

Example: Let f be continuous function on $[a, b]$. If $I = [c, d]$ let $u: I \rightarrow \mathbb{R}$ be differentiable function on $[c, d]$ and satisfy $u(I) \subseteq [a, b]$.

If $G(x) = \int_a^{u(x)} f(t) dt$, for $x \in I$. Find $G'(x)$ for $x \in I$.

Solution: We can write

$$G(x) = \int_a^{u(x)} f(t) dt = (h \circ u)(x),$$

where $h(x) = \int_a^x f(t) dt$.

Now if we apply the chain rule, we have

$$G'(x) = h'(u(x)) u'(x) = f(u(x)) u'(x).$$

Exercise: Find $F'(x)$, when F is defined on $[0, 1]$ as follows:

a) $F(x) = \int_0^x \sin t^2 dt$

b) $F(x) = \int_0^{x^2} (1 + t^3)^{-1} dt$

c) $F(x) = \int_{x^2}^x \sqrt{1 + t^2} dt$

d) $F(x) = \int_0^{\sin x} \cos t dt$.

① $u = 0 \quad u' = 0$
 $\sin t \quad t' = 1$
 $f(t) = \sin t^2$

$f(u) = \sin u^2 = \sin x^2$
 $f(x) = f(u)$

5- Methods of Integration

Theorem: Mean Value Theorem for Integrals

Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous real-valued function on $[a, b]$. Then, there exists $c \in [a, b]$ such that

$$\int_a^b f = f(c)(b - a).$$

Proof. Let $F(x) = \int_a^x f$. Since f is continuous on $[a, b]$, $F'(x) = f(x)$, for all $x \in [a, b]$. Thus, by the mean value theorem, there exists $c \in [a, b]$ such that

$$\int_a^b f(x) dx = F(b) - F(a) = F'(c)(b - a) = f(c)(b - a).$$

Theorem: Substitution Theorem

Let $\varphi: [a, b] \rightarrow \mathbb{R}$ be a deferrable on $[a, b]$ and $\varphi' \in \mathcal{R}([a, b])$. If f is continuous on an interval $I = \varphi([a, b])$, then

$$\int_a^b \underbrace{f(\varphi(t))}_{\text{دالين صغها به جي ليجن}} \underbrace{\varphi'(t)}_{\text{صغتها}} dt = \int_{\varphi(a)}^{\varphi(b)} f(x) dx.$$

Proof.

For $x \in I$ define

$$F(x) = \int_{\varphi(a)}^x f(t) dt.$$

Since f is continuous, $F'(x) = f(x)$ for all $x \in I$. By the chain rule

$$\frac{d}{dt} F(\varphi(t)) = F'(\varphi(t)) \varphi'(t) = f(\varphi(t)) \varphi'(t), \quad \forall t \in I.$$

Therefore,

$$\int_a^b f(\varphi(t)) \varphi'(t) dt = [F(\varphi(t))]_a^b = F(\varphi(b)) - F(\varphi(a)) = \int_{\varphi(a)}^{\varphi(b)} f(x) dx.$$

Example: Evaluate $\int_0^1 (1 + 2x)^2 dx$.

Solution: Let $u = 1 + 2x \Rightarrow \frac{du}{dx} = 2 \Rightarrow dx = \frac{1}{2} du$, when $x = 0 \Rightarrow u = 1$, and when $x = 1 \Rightarrow u = 3$. Then,

$$\int_0^1 (1 + 2x)^2 dx = \frac{1}{2} \int_1^3 u^2 du = \frac{1}{2} \left[\frac{u^3}{3} \right]_1^3 = \frac{1}{2} \left(\frac{3^3}{3} - \frac{1}{3} \right) = \frac{8}{6}.$$

Example: Evaluate $\int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}} dt$.

Solution: Let $u = 1 + \sqrt{t} \Rightarrow \frac{du}{dt} = \frac{1}{2\sqrt{t}} \Rightarrow dt = 2\sqrt{t} du$, when $t = 1 \Rightarrow u = 2$, and when $t = 4 \Rightarrow u = 3$. Then,

$$\begin{aligned} \int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}} dt &= \int_2^3 \frac{\sqrt{u}}{\sqrt{t}} 2\sqrt{t} du = 2 \int_2^3 \sqrt{u} du = 2 \int_2^3 u^{\frac{1}{2}} du = \frac{4}{3} u^{\frac{3}{2}} \Big|_2^3 \\ &= \frac{4}{3} (3\sqrt{3} - 2\sqrt{2}). \end{aligned}$$



Example: Evaluate $\int_0^2 \frac{t}{1+t^2} dt$.

Solution: Let $u = 1 + t^2 \Rightarrow \frac{du}{dt} = 2t \Rightarrow dt = \frac{1}{2t} du$, when $t = 0 \Rightarrow u = 1$, and when $t = 2 \Rightarrow u = 5$. Then,

$$\int_0^2 \frac{t}{1+t^2} dt = \frac{1}{2} \int_1^5 \frac{1}{u} du = \frac{1}{2} \ln u \Big|_1^5 = \frac{1}{2} \ln 5.$$

Exercise: Evaluate the following integrals: justify each step

a) $\int_0^1 t\sqrt{1+t^2} dt$

b) $\int_0^1 t^2\sqrt{1+t^3} dt$

c) $\int_0^3 \frac{t}{\sqrt{1+t^2}} dt$

Handwritten notes for exercise c):
 $u = 1+t^2 \Rightarrow \frac{du}{dt} = 2t \Rightarrow \frac{du}{2t} = dt$
 $t=0 \Rightarrow u=1$
 $t=3 \Rightarrow u=10$
 $\int_0^3 \frac{t}{\sqrt{1+t^2}} dt = \int_1^{10} \frac{1}{\sqrt{u}} \frac{du}{2} = \frac{1}{2} \int_1^{10} u^{-1/2} du = \frac{1}{2} [2\sqrt{u}]_1^{10} = \sqrt{10} - \sqrt{1}$

Theorem: Integration by Parts

Let f, g be deferrable functions on $[a, b]$ and have antiderivatives f', g' on $[a, b]$ then,

$$\int_a^b f(x)g'(x)dx = f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x)dx.$$

Proof.

Since $(fg)' = f'g + fg'$ and the function $(fg)'$ is integrable on $[a, b]$ then,

$$f(b)g(b) - f(a)g(a) = \int_a^b (fg)' = \int_a^b f'g + \int_a^b fg'.$$

Thus,

$$\int_a^b f(x)g'(x)dx = f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x)dx.$$

Example: Evaluate $\int_1^e x \ln x dx$.

Solution:

$$\begin{aligned} \int_1^e x \ln x dx &= \int_1^e \left(\frac{x^2}{2} \right)' \ln x dx = \frac{x^2}{2} \ln x \Big|_1^e - \int_1^e \frac{x^2}{2} (\ln x)' dx \\ &= \frac{e^2}{2} - \frac{1}{2} \int_1^e x dx = \frac{e^2}{2} - \frac{1}{2} \frac{e^2}{2} = \frac{e^2}{4}. \end{aligned}$$

Exercises: Evaluate the following integrals:

$$\textcircled{a} \int_1^e \frac{\ln x}{x} \frac{dx}{dx} \quad \textcircled{b} \int_0^{\frac{\pi}{4}} \tan^{-1}(x) dx \quad \textcircled{c} \int_0^{\pi} x^2 \cos x dx$$

$$\textcircled{d} \int_0^{\pi} e^x \sin x dx \quad \textcircled{e} \int_0^{\pi} e^x \cos x dx.$$

$\textcircled{a} \quad u = \ln x \quad du = \frac{1}{x} dx \quad dv = dx \quad v = x$
 $= x \ln x - \int_1^e x \cdot \frac{1}{x} dx$
 $= (e \ln e - \frac{1}{2} \ln(1)) - \frac{1}{2} \ln(e)$
 $= e - \frac{1}{2} - \frac{1}{2} = e - 1$
 $e - \frac{1}{2} + \frac{1}{2} = 1$

$\textcircled{b} \quad u = \tan^{-1} x \quad du = \frac{1}{1+x^2} dx \quad dv = dx \quad v = x$
 $= x \tan^{-1} x - \int_0^{\frac{\pi}{4}} x \cdot \frac{1}{1+x^2} dx$
 $= \left(\frac{\pi}{4} \tan^{-1} \frac{\pi}{4} - \frac{1}{2} \right) - \frac{1}{2} \left[\ln(1+x^2) \right]_0^{\frac{\pi}{4}}$

6-Improper Riemann integral

Definition3:(Improper Integral)

An integral is an improper integral:

- if either the interval of integration is not finite (improper integral of type 1)
- if the function to integrate is not continuous (not bounded) in the interval of integration (improper integral of type 2).

Examples:

1) $\int_0^{\infty} e^{-x} dx$ is an improper integral of type I since the upper limit of integration is infinite. $\int_0^{\infty} \frac{1}{e^x} dx$ improper type (1) $(0, \infty]$

2) $\int_0^1 x^{-1} dx$ is an improper integral of type II because $\frac{1}{x}$ is not continuous at 0. $\int_0^1 \frac{1}{x} dx$ $[0, 1]$ improper type (2)

3) $\int_0^{\infty} \frac{1}{x-1} dx$ is an improper integral of type 1 since the upper limit of integration is infinite. It is also an improper integral of type 2 because $\frac{1}{x-1}$ is not continuous at 1 and $1 \in [0, \infty)$. $\int_0^{\infty} \frac{1}{x-1} dx$ improper type (1) and (2)

4) $\int_{-2}^2 \frac{1}{x^2-1} dx$ is an improper integral of type 2 because $\frac{1}{x^2-1}$ is not continuous at $\{1, -1\}$. $\int_{-2}^2 \frac{1}{x^2-1} dx$ improper integral type (2)

5) $\int_0^{\pi} \tan(x) dx$ is an improper integral of type 2 because $\tan(x)$ is not continuous at $\frac{\pi}{2}$ and $\frac{\pi}{2} \in [0, \pi]$. $\int_0^{\pi} \tan(x) dx$ improper integral type (2)

Definition 3: Improper Riemann Integral of type I

1 – Let $f: [a, \infty) \rightarrow \mathbb{R}$. If $f \in \mathcal{R}([a, c])$ for any $c \in (a, \infty)$. The Improper Riemann integral of f on $[a, \infty)$ defined to be

$$\int_a^{\infty} f = \lim_{c \rightarrow \infty} \int_a^c f.$$

2 – Let $f: (-\infty, b] \rightarrow \mathbb{R}$. If $f \in \mathcal{R}([c, b])$ for any $c \in (-\infty, b)$. The Improper Riemann integral of f on $(-\infty, b]$ defined to be

$$\int_{-\infty}^b f = \lim_{c \rightarrow -\infty} \int_c^b f.$$

3 – Let $f: (-\infty, \infty) \rightarrow \mathbb{R}$. If $f \in \mathcal{R}([c, b])$ for any $c \in (-\infty, \infty)$. The Improper Riemann integral of f on $(-\infty, \infty)$ defined to be

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^a f + \int_a^{\infty} f = \lim_{c \rightarrow -\infty} \int_c^a f + \lim_{c \rightarrow \infty} \int_a^c f.$$

Definition 4: Improper Riemann Integral of type II

1 – Let $f: [a, b] \rightarrow \mathbb{R}$. If $f \in \mathcal{R}([c, b])$ for any $c \in (a, b)$. The Improper Riemann integral of f on $(a, b]$ defined to be

$$\int_a^b f = \lim_{c \rightarrow a^+} \int_c^b f. \quad [a, b]$$

2- Let $f: [a, b] \rightarrow \mathbb{R}$. If $f \in \mathcal{R}([c, b])$ for any $c \in (a, b)$. The Improper Riemann integral of f on $(a, b]$ defined to be

$$\int_a^b f = \lim_{c \rightarrow b^-} \int_a^c f.$$

3- Let $f: (a, b) \rightarrow \mathbb{R}$. If $f \in \mathcal{R}((a, b))$ for any $d \in (a, b)$. The Improper Riemann integral of f on (a, b) defined to be

$$\int_a^b f = \int_a^d f + \int_d^b f = \lim_{c \rightarrow a^+} \int_c^d f + \lim_{c \rightarrow b^-} \int_d^c f.$$

Theorem 3:

- If the limit exists as a finite number. In this case, the Improper Riemann integral $\int_a^b f$ is said to be convergent.
- Otherwise, $\int_a^b f$ is said to be divergent.

Theorem 4:

- The Improper Riemann integral $\int_0^1 \frac{1}{x^\alpha} dx$ converge if and only if $\alpha < 1$.
- The Improper Riemann integral $\int_1^\infty \frac{1}{x^\alpha} dx$ converge if and only if $\alpha > 1$.

Example: Study the convergence of the following integrations:

$$1) \int_1^\infty \frac{1}{x^2} dx \quad 2) \int_1^\infty \frac{1}{x} dx \quad 3) \int_1^\infty \frac{1}{e^x} dx \quad 4) \int_{-\infty}^\infty \frac{x}{1+x^2} dx$$

Solution:

$$1) \int_1^\infty \frac{1}{x^2} dx = \lim_{c \rightarrow \infty} \int_1^c \frac{1}{x^2} dx = \lim_{c \rightarrow \infty} \left[\frac{-1}{x} \right]_1^c = \lim_{c \rightarrow \infty} \left[\frac{-1}{c} + 1 \right] = 1$$

Then, $\int_1^\infty \frac{1}{x^2} dx$ converges. $= \frac{-1}{\infty} + 1 = 0 + 1 = 1$ (convergent)

2)

$$\int_1^\infty \frac{1}{x} dx = \lim_{c \rightarrow \infty} \int_1^c \frac{1}{x} dx = \lim_{c \rightarrow \infty} [\ln(x)]_1^c = \lim_{c \rightarrow \infty} [\ln(c)] = \infty$$

Then $\int_1^\infty \frac{1}{x} dx$ diverges.

3)

$$\int_1^\infty \frac{1}{e^x} dx = \lim_{c \rightarrow \infty} \int_1^c e^{-x} dx = \lim_{c \rightarrow \infty} \left[\frac{-1}{e^x} \right]_1^c = \lim_{c \rightarrow \infty} \left[\frac{-1}{e^c} + \frac{1}{e} \right] = \frac{1}{e}$$

Then $\int_1^\infty \frac{1}{e^x} dx$ converges.

4)

$$\begin{aligned} \int_{-\infty}^\infty \frac{x}{1+x^2} dx &= \int_{-\infty}^0 \frac{x}{1+x^2} dx + \int_0^\infty \frac{x}{1+x^2} dx \\ \int_{-\infty}^0 \frac{x}{1+x^2} dx &= \lim_{c \rightarrow -\infty} \int_c^0 \frac{x}{1+x^2} dx = \lim_{c \rightarrow -\infty} [\ln(1+x^2)]_c^0 \\ &= \lim_{c \rightarrow -\infty} [\ln(1) - \ln(1+c^2)] = -\infty. \end{aligned}$$

And

$$\int_0^{\infty} \frac{x}{1+x^2} dx = \lim_{c \rightarrow \infty} \int_0^c \frac{x}{1+x^2} dx = \lim_{c \rightarrow \infty} [\ln(1+x^2)]_0^c = \infty$$

Then $\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx$ diverges.

H. W: Discuss the convergence of the following integrations:

1) $\int_{-2}^3 \frac{1}{x^3} dx$

2) $\int_1^{\infty} \frac{1}{2x} dx$

3) $\int_1^{\infty} \frac{1}{e^x} dx$

4) $\int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx$

Exercise: Study the convergence of the following integrations:

1) $\int_{-3}^5 \frac{1}{x^2} dx$

2) $\int_1^{\infty} \frac{1}{2x} dx$

3) $\int_{-\infty}^0 \frac{1}{e^x} dx$

4) $\int_{-\infty}^0 x e^{-x^2} dx$

5) $\int_1^2 \frac{1}{x \ln x} dx$

6) $\int_0^1 \ln x dx$

7) $\int_0^1 x \ln x dx$

8) $\int_1^2 \frac{1}{\sqrt{x-1}} dx$

9) $\int_0^{\infty} e^{-x} dx$

10) $\int_2^{\infty} \frac{\ln x}{x} dx$

11) $\int_{-\infty}^{\infty} e^{-x} dx$

12) $\int_1^{\infty} \frac{1}{x^2} dx$

①
 $\int_{-2}^3 \frac{1}{x^3} dx$ $[-2, 3]$
 Improper integral type ②
 $\frac{1}{x^3}$ b/c $\frac{1}{x^3}$ is not
 Continuous at $x=0$

$\int_{-2}^3 \frac{1}{x^3} dx = \int_{-2}^0 \frac{1}{x^3} dx + \int_0^3 \frac{1}{x^3} dx$
 $= \lim_{c \rightarrow 0^-} \int_{-2}^c \frac{1}{x^3} dx + \lim_{c \rightarrow 0^+} \int_c^3 \frac{1}{x^3} dx$

note
 [if the part of the expression

diverges then the entire expression diverges]

$I = \lim_{c \rightarrow 0^-} \int_{-2}^c \frac{1}{x^3} dx$
 $\lim_{c \rightarrow 0^-} \left[-\frac{1}{2x^2} \right]_{-2}^c$

$\lim_{c \rightarrow 0^-} \left[-\frac{1}{2c^2} + \frac{1}{4} \right] = \infty$
 diverges

⑫ $\int_{-\infty}^{\infty} \frac{1}{\sqrt{3-x}} dx$ $(-\infty, \infty)$

Improper integral type ②
 $\lim_{c \rightarrow \infty} \int_c^{\infty} \frac{1}{\sqrt{3-x}} dx$

$\lim_{c \rightarrow \infty} \left[-2\sqrt{3-x} \right]_c^{\infty}$

$\lim_{c \rightarrow \infty} \left[2\sqrt{3} - 2\sqrt{3-c} \right]$

$\lim_{c \rightarrow \infty} \left[2\sqrt{3} - 2\sqrt{3-c} \right]$

$= -\infty$
 diverges