

Integration Techniques

I) Antiderivatives and Indefinite Integrals

Definition 1.1 A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x) \text{ for every } x \in I.$$

Example 1.1

(1) Let $F(x) = x^2 + 3x + 1$ and $f(x) = 2x + 3$.

Since $F'(x) = f(x)$, then the function $F(x)$ is an antiderivative of $f(x)$.

(2) Let $G(x) = \sin x + x$ and $g(x) = \cos x + 1$.

Since $G'(x) = \cos x + 1$, then the function $G(x)$ is an antiderivative of $g(x)$.

Definition 1.2 Let f be a continuous function on an interval I . The indefinite integral of f is the general antiderivative of f on I :

$$\int f(x) dx = F(x) + c.$$

The function f is called the integrand, the symbol $\int$ is the integral sign, x is called the variable of the integration and c is the constant of the integration.

Now, by using the previous definition, the general antiderivatives in Example 1.1 are

$$1 \quad \int (2x + 3) dx = x^2 + 3x + c.$$

$$2 \quad \int (\cos x + 1) dx = \sin x + x + c.$$

Basic Integration Rules

■ **Rule 1:** Power of x .

$$\frac{d}{dx} x^{n+1} = (n+1)x^n, \text{ so } \int (n+1)x^n dx = x^{n+1} + c$$

Generally, for $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c.$$

In words, to integrate the function x^n , we add 1 to the power and divide the function by $n+1$. If $n = -1$, we have a special case

$$\int 1 dx = x + c.$$

Rule 2: Trigonometric functions.

$$\frac{d}{dx} \sin x = \cos x, \text{ so } \int \cos x \, dx = \sin x + c$$
$$\frac{d}{dx} \cos x = -\sin x, \text{ so } \int -\sin x \, dx = \cos x + c$$

Therefore, $\int \sin x \, dx = -\cos x + c$.

The other trigonometric functions with the previous rules are listed in the following table:

Derivative	Indefinite Integral
$\frac{d}{dx}(x) = 1$	$\int 1 \, dx = x + c$
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x \, dx = \sin x + c$
$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x \, dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + c$ ✓
$\frac{d}{dx}(-\cot x) = \csc^2 x$	$\int \csc^2 x \, dx = -\cot x + c$ ✓
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + c$ ✓
$\frac{d}{dx}(-\csc x) = \csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + c$ ✓

Example 1.4 Evaluate the integral.

(1) $\int x^{-3} \, dx$

(2) $\int \frac{1}{\cos^2 x} \, dx$

Solution:

(1) $\int x^{-3} \, dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$.

(2) $\int \frac{1}{\cos^2 x} \, dx = \int \sec^2 x \, dx = \tan x + c$.

$(\sec x = \frac{1}{\cos x} \Rightarrow \sec^2 x = \frac{1}{\cos^2 x})$

Theorem 1.2 Assume f and g have antiderivatives on an interval I , then

1. $\frac{d}{dx} \int f(x) dx = f(x).$
2. $\int \frac{d}{dx}(F(x)) dx = F(x) + c.$
3. $\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx.$
4. $\int k f(x) dx = k \int f(x) dx$, where k is a constant.

Example 1.5 Evaluate the integral.

- (1) $\int (4x + 3) dx$
- (2) $\int (2 \sin x + 3 \cos x) dx$
- (3) $\int (\sqrt{x} + \sec^2 x) dx$
- (4) $\int \frac{d}{dx}(\sin x) dx$
- (5) $\frac{d}{dx} \int \sqrt{x+1} dx$

Solution:

- (1) $\int (4x + 3) dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$
- (2) $\int (2 \sin x + 3 \cos x) dx = -2 \cos x + 3 \sin x + c.$
- (3) $\int (\sqrt{x} + \sec^2 x) dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \tan x + c = \frac{2x^{\frac{3}{2}}}{3} + \tan x + c.$
- (4) $\int \frac{d}{dx}(\sin x) dx = \sin x + c.$
- (5) $\frac{d}{dx} \int \sqrt{x+1} dx = \sqrt{x+1}.$

Example 1.6 If $\int f(x) dx = x^2 + c_1$ and $\int g(x) dx = \tan x + c_2$, find $\int (3f(x) - 2g(x)) dx$.

Solution:

From the third and fourth properties, $\int (3f(x) - 2g(x)) dx = 3 \int f(x) dx - 2 \int g(x) dx = 3x^2 - 2 \tan x + c$, where $c = 3c_1 - 2c_2$.

II) Integration by Substitution

Theorem 1.3 Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on the interval J contains the range of the function g . If F is an antiderivative of the function f on J , then

$$\int f(g(x))g'(x) dx = F(g(x)) + c, \quad x \in I.$$

Example 1.10 Evaluate the integral $\int 2x(x^2 + 1)^3 dx$.

Solution:

We can use the previous theorem as follows:

let $f(x) = x^3$ and $g(x) = x^2 + 1$, then $f(g(x)) = (x^2 + 1)^3$. Since $g'(x) = 2x$, then from Theorem 1.3, we have

$$\int 2x(x^2 + 1)^3 dx = \frac{(x^2 + 1)^4}{4} + c.$$

Example 1.12 Evaluate the integral $\int \frac{x^2 - 1}{(x^3 - 3x + 1)^6} dx$.

Solution:

Let $u = x^3 - 3x + 1$, then $du = 3(x^2 - 1) dx$. By substitution, we have

$$\frac{1}{3} \int u^{-6} du = \frac{1}{3} \frac{1}{-5u^5} + c = \frac{-1}{15(x^3 - 3x + 1)^5} + c.$$

Corollary 1.1 If $\int f(x) dx = F(x) + c$, then for any $a \neq 0$,

$$\int f(ax \pm b) dx = \frac{1}{a} F(ax \pm b) + c.$$

Example 1.13 Evaluate the integral.

(1) $\int \sqrt{2x-5} \, dx$

(2) $\int \cos (3x+4) \, dx$

Solution:

From Corollary 1.1, we have

(1) $\int \sqrt{2x-5} \, dx = \frac{1}{2} \frac{(2x-5)^{3/2}}{3/2} + c = \frac{(2x-5)^{3/2}}{3} + c.$

(2) $\int \cos (3x+4) \, dx = \frac{1}{3} \sin (3x+4) + c.$

III) Integration by Parts

Integration by parts is a method to transfer the original integral to an easier one that can be evaluated. Practically, the integration by parts divides the original integral into two parts u and dv , then we find the du by deriving u and v by integrating dv .

Theorem 5.1 If $u = f(x)$ and $v = g(x)$ such that f' and g' are continuous, then

$$\int u \, dv = uv - \int v \, du.$$

Example 5.1 Evaluate the integral $\int x \cos x \, dx$.

Solution:

Let $I = \int x \cos x \, dx$. Let $u = x$ and $dv = \cos x \, dx$. Hence,

$$u = x \Rightarrow du = dx,$$

$$dv = \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x.$$

From Theorem 5.1, we have

$$I = x \sin x - \int \sin x \, dx = x \sin x + \cos x + c.$$

Try to choose

$u = \cos x$ and $dv = x \, dx$

Do you have the same result?

Example 5.2 Evaluate the integral $\int x e^x \, dx$.

Solution:

Let $I = \int x e^x \, dx$. Let $u = x$ and $dv = e^x \, dx$. Hence,

$$u = x \Rightarrow du = dx,$$

$$dv = e^x \, dx \Rightarrow v = \int e^x \, dx = e^x.$$

From Theorem 5.1, we have

$$I = x e^x - \int e^x \, dx = x e^x - e^x + c.$$

Try to choose $u = e^x$ and $dv = x \, dx$

We will obtain

$$I = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x \, dx.$$

However, the integral $\int \frac{x^2}{2} e^x \, dx$ is more difficult than the original one $\int x e^x \, dx$.

Example 5.3 Evaluate the integral $\int \ln x \, dx$.

Solution: Let $I = \int \ln x \, dx$. Let $u = \ln x$ and $dv = dx$. Hence,

$$\begin{aligned}u &= \ln x \Rightarrow du = \frac{1}{x} \, dx, \\dv &= dx \Rightarrow v = \int 1 \, dx = x.\end{aligned}$$

From Theorem 5.1, we obtain $I = x \ln x - \int x \frac{1}{x} \, dx = x \ln x - \int 1 \, dx = x \ln x - x + c$.

Example 5.4 Evaluate the integral $\int e^x \cos x \, dx$.

Solution: Let $I = \int e^x \cos x \, dx$. Let $u = e^x$ and $dv = \cos x \, dx$.

$$\begin{aligned}u &= e^x \Rightarrow du = e^x \, dx, \\dv &= \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x.\end{aligned}$$

Hence, $I = e^x \sin x - \int e^x \sin x \, dx$.

The integral $\int e^x \sin x \, dx$ cannot be evaluated. Therefore, we use the integration by parts again where we assume $J = \int e^x \sin x \, dx$. Let $u = e^x$ and $dv = \sin x \, dx$. Hence,

$$\begin{aligned}u &= e^x \Rightarrow du = e^x \, dx, \\dv &= \sin x \, dx \Rightarrow v = \int \sin x \, dx = -\cos x.\end{aligned}$$

Hence, $J = -e^x \cos x + \int e^x \cos x \, dx$. By substituting the result of J into I , we have

$$\begin{aligned}I &= e^x \sin x - J \\&= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx\end{aligned}$$

This implies

$$2I = e^x \sin x + e^x \cos x \Rightarrow I = \frac{1}{2}(e^x \sin x + e^x \cos x) \Rightarrow \int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + c.$$

Example 5.6 Evaluate the integral $\int_0^1 \tan^{-1} x \, dx$.

Solution:

Let $I = \int \tan^{-1} x \, dx$. Let $u = \tan^{-1} x$ and $dv = dx$. Hence,

$$u = \tan^{-1} x \Rightarrow du = \frac{1}{x^2 + 1} dx,$$

$$dv = dx \Rightarrow v = \int 1 \, dx = x.$$

By applying Theorem 5.1, we obtain

$$I = x \tan^{-1} x - \int \frac{x}{x^2 + 1} dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + c.$$

Therefore,

$$\int_0^1 \tan^{-1} x \, dx = \left[x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) \right]_0^1 = (\tan^{-1}(1) - \frac{1}{2} \ln 2) - (0 - \frac{1}{2} \ln 1) = \frac{\pi}{4} - \ln \sqrt{2}.$$

VI) Integrals of Rational Functions

In this section, we study rational functions of form $q(x) = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials. The previous techniques are not suitable to evaluate some integrals that consist of rational functions. Therefore, we need to introduce a new technique to integrate the rational functions. This technique is called decomposition of rational functions into a sum of partial fractions.

The practical steps to evaluate integrals of the rational functions can be summarized as follows:

- **Step 1:** If the degree of $g(x)$ is less than the degree of $f(x)$, we do polynomial long-division; otherwise we move to step 2.

From the long division shown on the right side, we have

$$q(x) = \frac{f(x)}{g(x)} = h(x) + \frac{r(x)}{g(x)},$$

where $h(x)$ is the quotient and $r(x)$ is the remainder.

$$\begin{array}{r} \overline{) \begin{array}{l} h(x) \\ f(x) \\ \dots \\ \dots \\ r(x) \end{array}} \\ \underline{ } \end{array}$$

- **Step 2:** Factor the denominator $g(x)$ into irreducible polynomials where the factors are either linear or irreducible quadratic polynomials.¹
- **Step 3:** Find the partial fraction decomposition. This step depends on the result of step 2 where the fraction $\frac{f(x)}{g(x)}$ or $\frac{r(x)}{g(x)}$ can be written as a sum of partial fractions:

$$q(x) = P_1(x) + P_2(x) + P_3(x) + \dots + P_n(x),$$

each $P_k(x) = \frac{A_k}{(ax+b)^n}, n \in \mathbb{N}$ or $P_k(x) = \frac{A_kx+B_k}{(ax^2+bx+c)^n}$ if $b^2 - 4ac < 0$. The constants A_k and B_k are real numbers and computed later.

- **Step 4:** Integrate the result of step 3.

Example 5.12 Evaluate the integral $\int \frac{x+1}{x^2-2x-8} dx$.

Solution:

Step 1: This step can be skipped since the degree of $f(x) = x+1$ is less than the degree of $g(x) = x^2 - 2x - 8$.

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

$$g(x) = x^2 - 2x - 8 = (x+2)(x-4).$$

Step 3: Find the partial fraction decomposition.

$$\frac{x+1}{x^2-2x-8} = \frac{A}{x+2} + \frac{B}{x-4} = \frac{Ax-4A+Bx+2B}{(x+2)(x-4)}.$$

We need to find the constants A and B .

Coefficients of the numerators:

$$\begin{aligned} A+B &= 1 \rightarrow \textcircled{1} \\ -4A+2B &= 1 \rightarrow \textcircled{2} \end{aligned}$$

By doing some calculation, we obtain $A = \frac{1}{6}$ and $B = \frac{5}{6}$.

Illustration

Multiply equation $\textcircled{1}$ by 4 and add the result to equation $\textcircled{2}$

$$\begin{array}{r} 4A+4B=4 \\ -4A+2B=1 \\ \hline 6B=5 \end{array}$$

Step 4: Integrate the result of step 3.

$$\int \frac{x+1}{x^2-2x-8} dx = \int \frac{1/6}{x+2} dx + \int \frac{5/6}{x-4} dx = \frac{1}{6} \ln|x+2| + \frac{5}{6} \ln|x-4| + c.$$

Example 5.13 Evaluate the integral $\int \frac{2x^3-4x^2-15x+5}{x^2+3x+2} dx$.

Solution:

Step 1: Do the polynomial long-division.

Since the degree of the denominator $g(x)$ is less than the degree of the numerator $f(x)$, we do the polynomial long-division given on the right side. Then, we have

$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2}.$$

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

$$g(x) = x^2 + 3x + 2 = (x+1)(x+2).$$

Step 3: Find the partial fraction decomposition.

$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2} = (2x-10) + \frac{A}{x+1} + \frac{B}{x+2} = (2x-10) + \frac{Ax+2A+Bx+B}{(x+1)(x+2)}.$$

We need to find the constants A and B .

Coefficients of the numerators:

$$A + B = 11 \rightarrow \textcircled{1}$$

$$2A + B = 25 \rightarrow \textcircled{2}$$

By doing some calculation, we have $A = 14$ and $B = -3$.

Illustration

$$-2 \times \textcircled{1} + \textcircled{2}$$

$$-2A - 2B = -22$$

$$2A + B = 25$$

$$-----$$

$$-B = 3$$

Step 4: Integrate the result of step 3.

$$\begin{aligned}\int q(x) dx &= \int (2x - 10) dx + \int \frac{14}{x+1} dx + \int \frac{-3}{x+2} dx \\ &= x^2 - 10x + 14 \ln |x+1| - 3 \ln |x+2| + c.\end{aligned}$$

Example 5.14 Evaluate the integral $\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} dx$.

Solution:

Steps 1 and 2 can be skipped in this example.

Step 3: Find the partial fraction decomposition.

Since the denominator $g(x)$ has repeated factors, then

$$\frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-5} = \frac{A(x^2 - 4x - 5) + B(x-5) + C(x^2 + 2x + 1)}{(x+1)^2(x-5)}.$$

Coefficients of the numerators:

$$A + C = 2 \rightarrow \textcircled{1}$$

$$-4A + B + 2C = -25 \rightarrow \textcircled{2}$$

$$-5A - 5B + C = -33 \rightarrow \textcircled{3}$$

Illustration

$$5 \times \textcircled{2} + \textcircled{3} =$$

$$-25A + 11C = -158 \rightarrow \textcircled{4}$$

$$25 \times \textcircled{1} + \textcircled{4} =$$

$$36C = -108 \Rightarrow C = -3$$

By solving the system of equations, we have $A = 5$, $B = 1$ and $C = -3$.

Step 4: Integrate the result of step 3.

$$\begin{aligned}\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} dx &= \int \frac{5}{x+1} dx + \int \frac{1}{(x+1)^2} dx + \int \frac{-3}{x-5} dx \\ &= 5 \ln |x+1| + \int (x+1)^{-2} dx - 3 \ln |x-5| \\ &= 5 \ln |x+1| - \frac{1}{(x+1)} - 3 \ln |x-5| + c.\end{aligned}$$

Example 5.15 Evaluate the integral $\int \frac{x+1}{x(x^2+1)} dx$.

Solution:

Steps 1 and 2 can be skipped in this example.

Step 3: Find the partial fraction decomposition.

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}.$$

Coefficients of the numerators:

$$A+B=0 \rightarrow \textcircled{1}$$

$$C=1 \rightarrow \textcircled{2}$$

$$A=1 \rightarrow \textcircled{3}$$

We have $A=1$, $B=-1$ and $C=1$.

Step 4: Integrate the result of step 3.

$$\begin{aligned} \int \frac{x+1}{x(x^2+1)} dx &= \int \frac{1}{x} dx + \int \frac{-x+1}{x^2+1} dx \\ &= \ln|x| - \int \frac{x}{x^2+1} dx + \int \frac{1}{x^2+1} dx \\ &= \ln|x| - \frac{1}{2} \ln(x^2+1) + \tan^{-1} x + c. \end{aligned}$$

V) Integration of Powers of Trigonometric Functions

Example 5.7 Evaluate the integral.

$$(1) \int \sin^3 x \, dx$$

$$(3) \int \sin^5 x \cos^4 x \, dx$$

$$(2) \int \cos^4 x \, dx$$

$$(4) \int \sin^2 x \cos^2 x \, dx$$

Solution:

(1) Write $\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x$. Hence,

$$\int \sin^3 x \, dx = \int (1 - \cos^2 x) \sin x \, dx.$$

Let $u = \cos x$, then $du = -\sin x \, dx$. By substitution, we have

$$-\int (1 - u^2) \, du = -u + \frac{u^3}{3} + c.$$

This implies

$$\int \sin^3 x \, dx = -\cos x + \frac{1}{3} \cos^3 x + c.$$

(2) Write $\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$. Hence,

$$\begin{aligned} \int \cos^4 x \, dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \frac{1}{4} \int (1 + 2\cos 2x + \cos^2 2x) \, dx \\ &= \frac{1}{4} \left(\int 1 \, dx + \int 2\cos 2x \, dx + \int \cos^2 2x \, dx \right) \\ &= \frac{1}{4} \left(x + \sin 2x + \frac{1}{2} \int (1 + \cos 4x) \, dx \right) \\ &= \frac{1}{4} \left(x + \sin 2x + \frac{1}{2} \left(x + \frac{\sin 4x}{4} \right) \right) + c. \end{aligned}$$

(3) Write $\sin^5 x \cos^4 x = \sin^4 x \cos^4 x \sin x = (1 - \cos^2 x)^2 \cos^4 x \sin x$.

Let $u = \cos x$, then $du = -\sin x \, dx$. Thus, the integral becomes

$$-\int (1 - u^2)^2 u^4 \, du = -\int (u^4 - 2u^6 + u^8) \, du = -\left(\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right) + c.$$

This implies $\int \sin^5 x \cos^4 x \, dx = -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + c$.

(4) The integrand $\sin^2 x \cos^2 x = \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) = \frac{1-\cos^2 2x}{4} = \frac{\sin^2 2x}{4} = \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)$. Hence,

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{8} \int (1 - \cos 4x) \, dx = \frac{1}{8} \left(x - \frac{\sin 4x}{4} \right) + c.$$

Integration of Forms $\sin ux \cos vx$, $\sin ux \sin vx$ and $\cos ux \cos vx$

We deal with these integrals by using the following formulas:

$$\sin ux \cos vx = \frac{1}{2} (\sin (u-v)x + \sin (u+v)x)$$

$$\sin ux \sin vx = \frac{1}{2} (\cos (u-v)x - \cos (u+v)x)$$

$$\cos ux \cos vx = \frac{1}{2} (\cos (u-v)x + \cos (u+v)x)$$

Example 5.10 Evaluate the integral.

(1) $\int \sin 5x \sin 3x \, dx$

(3) $\int \cos 5x \sin 2x \, dx$

(2) $\int \sin 7x \cos 2x \, dx$

(4) $\int \cos 6x \cos 4x \, dx$

Solution:

(1) From the previous formulas, we have $\sin 5x \sin 3x = \frac{1}{2} (\cos 2x - \cos 8x)$. Hence,

$$\begin{aligned} \int \sin 5x \sin 3x \, dx &= \frac{1}{2} \int (\cos 2x - \cos 8x) \, dx \\ &= \frac{1}{4} \sin 2x - \frac{1}{16} \sin 8x + c. \end{aligned}$$

(2) Since $\sin 7x \cos 2x = \frac{1}{2} (\sin 5x + \sin 9x)$, then

$$\begin{aligned} \int \sin 7x \cos 2x \, dx &= \frac{1}{2} \int (\sin 5x + \sin 9x) \, dx \\ &= -\frac{1}{10} \cos 5x - \frac{1}{18} \cos 9x + c. \end{aligned}$$

(3) Since $\cos 5x \sin 2x = \frac{1}{2} (\sin 3x + \sin 7x)$, then

$$\begin{aligned} \int \cos 5x \sin 2x \, dx &= \frac{1}{2} \int (\sin 3x + \sin 7x) \, dx \\ &= -\frac{1}{6} \cos 3x - \frac{1}{14} \cos 7x + c. \end{aligned}$$

(4) Since $\cos 6x \cos 4x = \frac{1}{2} (\cos 2x + \cos 10x)$, then

$$\begin{aligned} \int \cos 6x \cos 4x \, dx &= \frac{1}{2} \int (\cos 2x + \cos 10x) \, dx \\ &= \frac{1}{4} \sin 2x + \frac{1}{20} \sin 10x + c. \end{aligned}$$

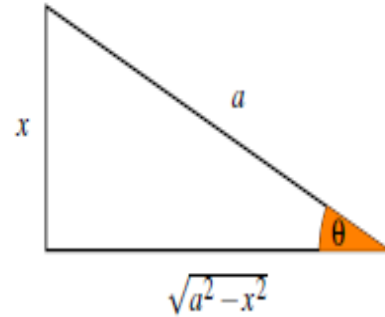
VI) Trigonometric Substitutions

In this section, we are going to study integrals containing the following expressions $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$ where $a > 0$. To get rid of the square roots, we convert them using substitutions involving trigonometric functions. In the following, we explain the conversion of the square roots:

■ $\sqrt{a^2 - x^2} = a \cos \theta$ if $x = a \sin \theta$.

If $x = a \sin \theta$ where $\theta \in [-\pi/2, \pi/2]$, then

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= \sqrt{a^2(1 - \sin^2 \theta)} \\ &= \sqrt{a^2 \cos^2 \theta} \\ &= a \cos \theta.\end{aligned}$$

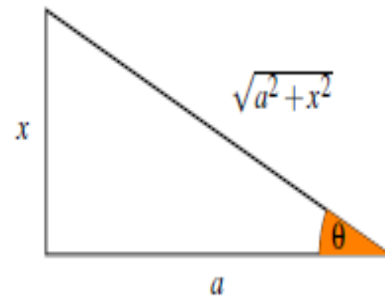


If the expression $\sqrt{a^2 - x^2}$ is in a denominator, then we assume $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

■ $\sqrt{a^2 + x^2} = a \sec \theta$ if $x = a \tan \theta$.

If $x = a \tan \theta$ where $\theta \in (-\pi/2, \pi/2)$, then

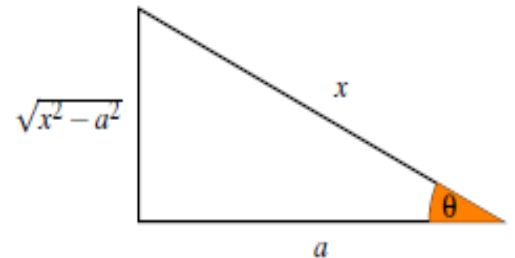
$$\begin{aligned}\sqrt{a^2 + x^2} &= \sqrt{a^2 + a^2 \tan^2 \theta} \\ &= \sqrt{a^2(1 + \tan^2 \theta)} \\ &= \sqrt{a^2 \sec^2 \theta} \\ &= a \sec \theta.\end{aligned}$$



■ $\sqrt{x^2 - a^2} = a \tan \theta$ if $x = a \sec \theta$.

If $x = a \sec \theta$ where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$, then

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2 \sec^2 \theta - a^2} \\ &= \sqrt{a^2(\sec^2 \theta - 1)} \\ &= \sqrt{a^2 \tan^2 \theta} \\ &= a \tan \theta.\end{aligned}$$



The previous discussion can be summarized in the following table:

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

Example 5.11 Evaluate the integral.

$$(1) \int \frac{x^2}{\sqrt{1-x^2}} dx$$

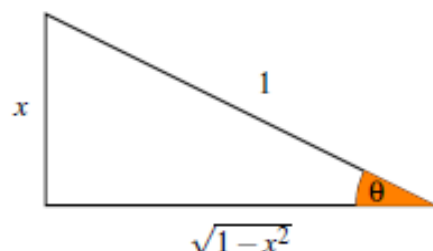
$$(2) \int_5^6 \frac{\sqrt{x^2-25}}{x^4} dx$$

$$(3) \int \sqrt{x^2+9} dx$$

Solution:

(1) Let $x = \sin \theta$ where $\theta \in (-\pi/2, \pi/2)$, thus $dx = \cos \theta d\theta$. By substitution, we have

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{1-x^2}} dx &= \int \frac{\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta \\
 &= \int \frac{\sin^2 \theta \cos \theta}{\cos \theta} d\theta \\
 &= \int \sin^2 \theta d\theta \\
 &= \frac{1}{2} \int (1 - \cos 2\theta) d\theta \\
 &= \frac{1}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) + c \\
 &= \frac{1}{2} (\theta - \sin \theta \cos \theta) + c.
 \end{aligned}$$

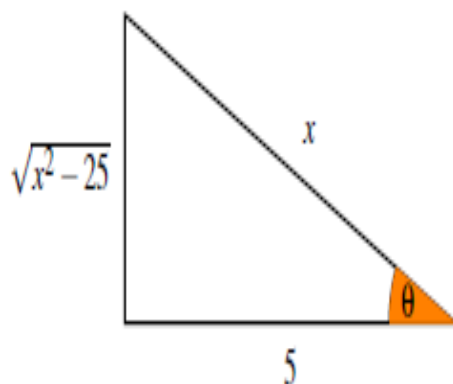


Now, we must return to the original variable x :

$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \frac{1}{2} (\sin^{-1} x - x\sqrt{1-x^2}) + c.$$

(2) Let $x = 5 \sec \theta$ where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$, thus $dx = 5 \sec \theta \tan \theta d\theta$. After substitution, the integral becomes

$$\begin{aligned}
 \int \frac{\sqrt{25 \sec^2 \theta - 25}}{625 \sec^4 \theta} 5 \sec \theta \tan \theta d\theta &= \frac{1}{25} \int \frac{\tan^2 \theta}{\sec^3 \theta} d\theta \\
 &= \frac{1}{25} \int \sin^2 \theta \cos \theta d\theta \\
 &= \frac{1}{75} \sin^3 \theta + c.
 \end{aligned}$$



We must

return to the original variable x :

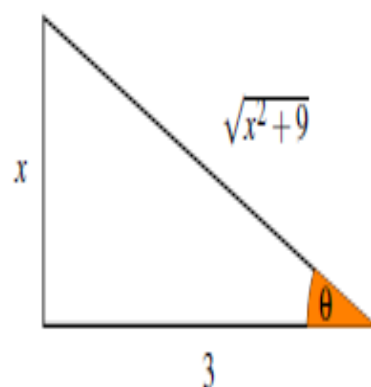
$$\int \frac{\sqrt{x^2 - 25}}{x^4} dx = \frac{(x^2 - 25)^{3/2}}{75x^3}$$

Hence,

$$\int_5^6 \frac{\sqrt{x^2 - 25}}{x^4} dx = \frac{1}{75} \left[\frac{(x^2 - 25)^{3/2}}{x^3} \right]_5^6 = \frac{1}{600}.$$

(3) Let $x = 3 \tan \theta$ where $\theta \in (-\pi/2, \pi/2)$. This implies $dx = 3 \sec^2 \theta d\theta$. By substitution, we have

$$\begin{aligned}
 \int \sqrt{x^2 + 9} dx &= \int \sqrt{9 \tan^2 \theta + 9} (3 \sec^2 \theta) d\theta \\
 &= 9 \int \sec^3 \theta d\theta \\
 &= \frac{9}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|).
 \end{aligned}$$



This implies

$$\int \sqrt{x^2 + 9} dx = \frac{9}{2} \left(\frac{x\sqrt{x^2 + 9}}{9} + \ln \left| \frac{\sqrt{x^2 + 9} + x}{3} \right| \right) + c.$$

VII) Improper Integrals

In this section, we deal with integrals over infinite intervals or with integrals that involve discontinuous integrands. In such cases, the integrals are called improper.

Definition 6.2 The integral $\int_a^b f(x) dx$ is called a proper integral if

1. the interval $[a, b]$ is finite and closed, and
2. $f(x)$ is defined on $[a, b]$.

If condition 1 or 2 is not satisfied, the integral is improper. In the following, we discuss the improper integrals.

Infinite Intervals

In this section, we study integrals of forms $\int_a^\infty f(x) dx$, $\int_{-\infty}^b f(x) dx$, $\int_{-\infty}^\infty f(x) dx$ where f is a continuous function.

Definition 6.3

1. Let f be a continuous function on $[a, \infty)$. The improper integral $\int_a^\infty f(x) dx$ is defined as follows:

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx \text{ if the limit exists.}$$

2. Let f be a continuous function on $(-\infty, b]$. The improper integral $\int_{-\infty}^b f(x) dx$ is defined as follows:

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx \text{ if the limit exists.}$$

The previous integrals are convergent (or to converge) if the limit exists as a finite number. However, if the limit does not exist or equals $\pm\infty$, the integral is called divergent (or to diverge).

3. Let f be a continuous function on $\mathbb{R}$ and $a \in \mathbb{R}$. The improper integral $\int_{-\infty}^\infty f(x) dx$ is defined as follows:

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx.$$

The integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.

Note:

1. If an improper integral is convergent, the value of the integral is the value of the limit.
2. If both integrals in item 3 converge, then the value of the improper integral is the sum of values of the two integrals.

Example 6.5 Determine whether the integral converges or diverges.

(1) $\int_0^{\infty} \frac{1}{(x+2)^2} dx$

(2) $\int_0^{\infty} \frac{x}{1+x^2} dx$

(3) $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

Solution:

(1) $\int_0^{\infty} \frac{1}{(x+2)^2} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{1}{(x+2)^2} dx.$
The integral

$$\int_0^t \frac{1}{(x+2)^2} dx = \int_0^t (x+2)^{-2} dx = \left[\frac{-1}{x+2} \right]_0^t = -\left(\frac{1}{t+2} - \frac{1}{2} \right).$$

Thus,

$$\lim_{t \rightarrow \infty} \int_0^t \frac{1}{(x+2)^2} dx = -\lim_{t \rightarrow \infty} \left(\frac{1}{t+2} - \frac{1}{2} \right) = -(0 - \frac{1}{2}) = \frac{1}{2}.$$

This implies that the integral converges and has the value $\frac{1}{2}$.

(2) $\int_0^{\infty} \frac{x}{1+x^2} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{x}{1+x^2} dx.$
The integral

$$\int_0^t \frac{x}{1+x^2} dx = \frac{1}{2} \left[\ln(1+x^2) \right]_0^t = \frac{1}{2} \ln(1+t^2) - \frac{1}{2} \ln(1) = \frac{1}{2} \ln(1+t^2).$$

Thus,

$$\lim_{t \rightarrow \infty} \int_0^t \frac{x}{1+x^2} dx = \frac{1}{2} \lim_{t \rightarrow \infty} \ln(1+t^2) = \infty.$$

The improper integral diverges.

(3) $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{1+x^2} dx + \lim_{t \rightarrow \infty} \int_0^t \frac{1}{1+x^2} dx.$

We know that $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$, so

$$\begin{aligned} \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{1+x^2} dx + \lim_{t \rightarrow \infty} \int_0^t \frac{1}{1+x^2} dx &= \lim_{t \rightarrow -\infty} [0 - \tan^{-1}(t)] + \lim_{t \rightarrow \infty} [\tan^{-1} t - 0] \\ &= -\lim_{t \rightarrow -\infty} \tan^{-1} t + \lim_{t \rightarrow \infty} \tan^{-1} t \\ &= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi. \end{aligned}$$

The integral is convergent and has the value π .

Discontinuous Integrands

Definition 6.4

1. If f is continuous on $[a, b)$ and has an infinite discontinuity at b i.e., $\lim_{x \rightarrow b^-} f(x) = \pm\infty$, then

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx \text{ if the limit exists.}$$

2. If f is continuous on $(a, b]$ and has an infinite discontinuity at a i.e., $\lim_{x \rightarrow a^+} f(x) = \pm\infty$, then

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx \text{ if the limit exists.}$$

In items 1 and 2, the integral is convergent if the limit exists as a finite number; otherwise the integral is divergent.

3. If f is continuous on $[a, b]$ except at $c \in (a, b)$ such that $\lim_{x \rightarrow c^\pm} f(x) = \pm\infty$, the improper integral $\int_a^b f(x) dx$ is defined as follows:

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

The integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.

Example 6.6 Determine whether the integral converges or diverges.

(1) $\int_0^4 \frac{1}{(4-x)^{3/2}} dx$

(2) $\int_0^{\pi/4} \frac{\cos x}{\sqrt{\sin x}} dx$

(3) $\int_{-3}^1 \frac{1}{x^2} dx$

Solution:

- (1) Since $\lim_{x \rightarrow 4^-} \frac{1}{(4-x)^{3/2}} = \infty$ and the integrand is continuous on $[0, 4)$, then from Definition 6.4,

$$\begin{aligned} \int_0^4 \frac{1}{(4-x)^{3/2}} dx &= \lim_{t \rightarrow 4^-} \int_0^t (4-x)^{-3/2} dx \\ &= \lim_{t \rightarrow 4^-} \left[\frac{2}{\sqrt{4-x}} \right]_0^t \\ &= \lim_{t \rightarrow 4^-} \left(\frac{2}{\sqrt{4-t}} - 1 \right) \\ &= \infty. \end{aligned}$$

Illustration

$$\begin{aligned} \int (4-x)^{-3/2} dx &= - \int -(4-x)^{-3/2} dx \\ &= 2(4-x)^{-1/2} + c \\ &= \frac{2}{\sqrt{4-x}} + c \end{aligned}$$

Thus, the integral diverges.

- (2) The limit $\lim_{x \rightarrow 0^+} \frac{\cos x}{\sqrt{\sin x}} = \infty$ and the integrand is continuous on $(0, \frac{\pi}{4}]$, thus

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}} dx &= \lim_{t \rightarrow 0^+} \int_t^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}} dx \\
&= 2 \lim_{t \rightarrow 0^+} \left[\sqrt{\sin x} \right]_t^{\frac{\pi}{4}} \\
&= 2 \lim_{t \rightarrow 0^+} \left(\frac{1}{\sqrt[4]{2}} - \sqrt{\sin t} \right) \\
&= \frac{2}{\sqrt[4]{2}}.
\end{aligned}$$

The integral converges and has the value $\frac{2}{\sqrt[4]{2}}$.

Illustration

$$\begin{aligned}
\int \frac{\cos x}{\sqrt{\sin x}} dx &= \int \cos x \sin^{-1/2} x dx \\
&= 2 \sin^{1/2} x + c
\end{aligned}$$

(3) Since $\lim_{x \rightarrow 0^-} \frac{1}{x^2} = \lim_{x \rightarrow 0^+} \frac{1}{x^2} = \infty$ and the integrand is continuous on $[-3, 0) \cup (0, 1]$, then

$$\begin{aligned}
\int_{-3}^1 \frac{1}{x^2} dx &= \int_{-3}^0 \frac{1}{x^2} dx + \int_0^1 \frac{1}{x^2} dx \\
&= \lim_{t \rightarrow 0^-} \int_{-3}^t \frac{1}{x^2} dx + \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{x^2} dx \\
&= -\lim_{t \rightarrow 0^-} \left[\frac{1}{x} \right]_{-3}^t - \lim_{t \rightarrow 0^+} \left[\frac{1}{x} \right]_t^1 \\
&= -\lim_{t \rightarrow 0^-} \left[\frac{1}{t} + \frac{1}{3} \right] - \lim_{t \rightarrow 0^+} \left[1 - \frac{1}{t} \right] \\
&= \infty.
\end{aligned}$$

The integral diverges.

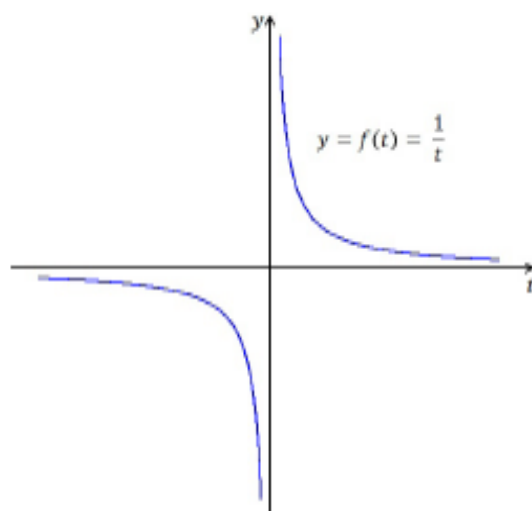


Figure 6.1