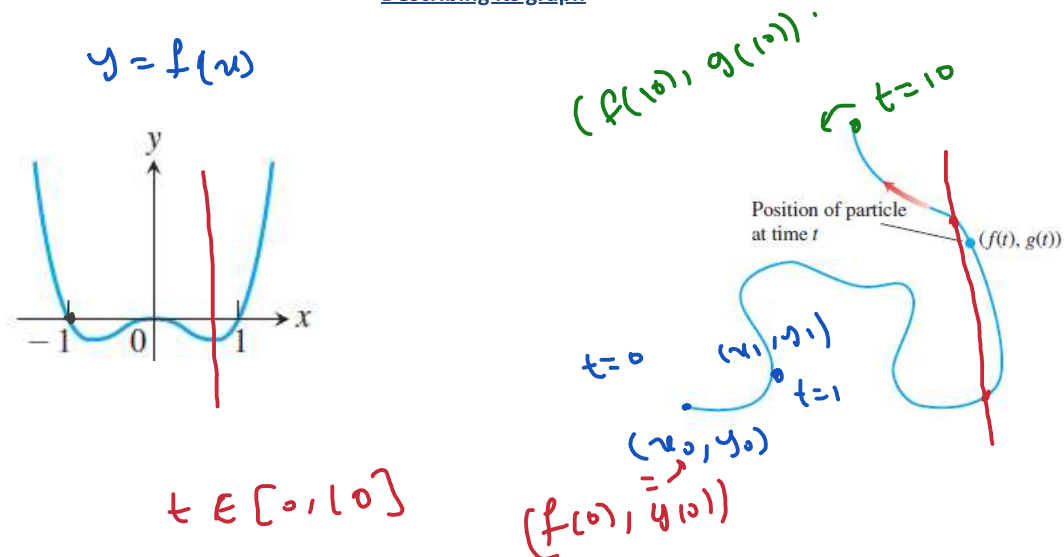


Math 203 (FA): 10.1 Parametrizations of Plane Curves

Saturday, January 24, 2026 2:27 PM

What is a Parametric Equation?

Describing its graph

$$\begin{array}{lcl}
 t_0 & \longrightarrow & x_0 \\
 t_1 & \longrightarrow & x_1 \\
 \vdots & & \vdots \\
 t & \longrightarrow & x_t = f(t) \\
 \text{input} & & \text{output}
 \end{array}
 \qquad
 \begin{array}{lcl}
 t_0 & \longrightarrow & y_0 \\
 t_1 & \longrightarrow & y_1 \\
 \vdots & & \vdots \\
 t & \longrightarrow & y_t = g(t)
 \end{array}$$

Feature	Function ($y = f(x)$)	Parametric ($x(t), y(t)$)
Dependency	y depends on x .	Both x and y depend on t .
Direction	No inherent "direction" of motion. <i>static data</i>	Has a direction of orientation (as t increases).
Flexibility	Limited (must pass Vertical Line Test).	Unlimited (can form loops, circles, etc.).
Physics	Shows "where" a curve is.	Shows "where" and "when" an object is there.

DEFINITION If x and y are given as functions

$$\underline{x = f(t)}, \quad y = g(t)$$

over an interval I of t -values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve. The equations are parametric equations for the curve.

an order pair $(x, y) = (f(t), g(t))$
 s.t. f & g are functions defined on t .
 input

$f(t), g(t)$ defined on t .

$$t \in I$$

closed and bounded

$$t \in [a, b] = I$$

initial value:
 $(f(a), g(a))$
 terminal value
 $(f(b), g(b))$

unbounded.

$$(-\infty, \infty) / [a, \infty)$$

$$(-\infty, a]$$

initial / terminal
 $(f(a), g(a))$

no initial / no terminal.
 has terminal
 $(f(a), g(a))$

EXAMPLE 2 Sketch the curve defined by the parametric equations

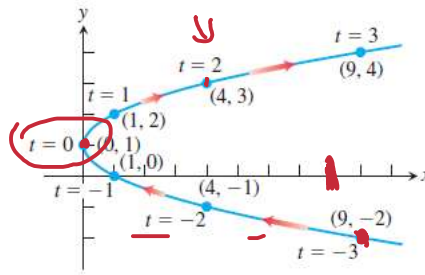
$$\underline{x = t^2}, \quad \underline{y = t + 1}, \quad -\infty < t < \infty.$$

Sol. Since x and y depends on t ,
 then $(x, y) = (t^2, t + 1)$ in $(-\infty, \infty)$

parametric equation.

time

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



upward to the right.

What eliminating the parameter means: to eliminate the parameter t , and obtain an algebraic equation for the curve in terms of x and y alone.

This helps us to sketch the curve of parametric equations

EXAMPLE 2 Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty.$$

Sol. Since $y = t + 1$ is a linear equation, then, we use it to find the

value of t :

$$t = y - 1. \quad \text{so,}$$

$$x = t^2 = (y - 1)^2 = y^2 - 2y + 1.$$

$$\text{so, } x = y^2 - 2y + 1$$

(1, 0)
(0, 1)

is a parabola $\hat{=}$ $a = 1 > 0$ that opens to

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$



the right since $a > 0$.

EXAMPLE 3 Graph the parametric curves

(a) $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$.

$I = [0, 2\pi]$

(b) $x = a \cos t$, $y = a \sin t$, $0 \leq t \leq 2\pi$.

Sol. (a) Since $I = [0, 2\pi]$, the curve has initial and terminal values:

(^{initial} $f(0)$, $g(0)$) = $(\cos(0), \sin(0)) = (1, 0)$

(^{terminal} $f(2\pi)$, $g(2\pi)$) = $(\cos(2\pi), \sin(2\pi)) = (1, 0)$

We use the identity:

$$\begin{aligned} 1 &= \cos^2 t + \sin^2 t \\ &= (\cos t)^2 + (\sin t)^2 \\ &= x^2 + y^2 \\ &= (x-0)^2 + (y-0)^2 \end{aligned}$$

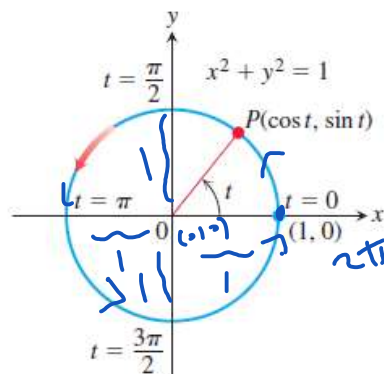
This is the equation of a circle with center $(0, 0)$ and radius 1.

Direction:

Counterclockwise.

$t=0$

$t=2\pi$



(b) initial value:

$$(f(0), g(0)) = (a \overset{1}{\cos(0)}, a \overset{0}{\sin(0)}) = (a, 0)$$

terminal value:

$$(f(2\pi), g(2\pi)) = (a \overset{1}{\cos(2\pi)}, a \overset{0}{\sin(2\pi)}) = (a, 0)$$

Now:

$$x = a \cos t, \quad y = a \sin t, \text{ then,}$$

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{a} = \sin t.$$

Now, we apply the identity!

$$1 = \cos^2 t + \sin^2 t$$

$$= (\cos t)^2 + (\sin t)^2$$

$$= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2$$

$$= \frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{x^2 + y^2}{a^2} \quad \text{[crossed out]} \quad \frac{1}{1}$$

nice #.

$$\text{So, } x^2 + y^2 = a^2$$

→ this is the equation of a circle with

center $(0, 0)$ and radius a . #



all

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